Math 33A
Linear Algebra and Applications

Midterm 2 — Lecture 1

Instructions: You have 50 minutes to complete this exam. There are 5 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. Please use a pen to write your solutions in the space provided below the statement of the problem, box your final answer, show all your work legibly, and clearly reference any results that you use. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Once the 50 minutes have elapsed, you are not allowed to continue writing and you are not allowed to communicate with anybody except the administrators of the exam. Please follow their requests at all times. Failure to comply with any of these instructions may have repercussions in your final grade.

Name: ______________________________

ID number: ___________________________

Section: _____________________________

Question	Points	Score
1	15	
2	20	
3	20	
4	25	
5	20	
Total:	100	

Problem 1. *15pts.*

Determine whether the following statements are true or false. If the statement is true, write **T** over the line provided before the statement. If the statement is false, write **F** over the line provided before the statement. Do NOT write "true" or "false".

(a) __**T**__ If A and B are symmetric $n \times n$ matrices, then $ABBA$ must be symmetric as well.

(b) __**F**__ If A is an invertible matrix such that $A^{-1} = A$, then A must be orthogonal.

(c) __**T**__ If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the inequality $\vec{x} \cdot (\text{proj}_V \vec{x}) \geq 0$ must hold.

(d) __**T**__ If matrix B is obtained by swapping two rows of an $n \times n$ matrix A, then the equation $\det(B) = -\det(A)$ must hold.

(e) __**F**__ There exist real invertible 3×3 matrices A and S such that $S^T AS = -A$.

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Problem 2. *20pts.*

Consider the vectors

$$\vec{u_1} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u_2} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u_3} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in $\mathbb{R}^4$. Is there a vector $\vec{u_4}$ in $\mathbb{R}^4$ such that $\mathfrak{B} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$ is an orthonormal basis? If there is, give $\mathfrak{B}$ explicitly.

Solution: Yes, there are two possible vectors

$$\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

which make $\mathfrak{B}$ into an orthonormal basis.

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Problem 3. *20pts.*

Find the QR factorization of the following matrix.

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Solution: Since the first matrix has orthonormal columns and the second matrix is upper triangular, we are almost given the QR factorization of M, we only need to fix the -4. When we multiply these two matrices, the second column of the first matrix will be multiplied by the second row of the second matrix. If we change the signs of this column and row, the product will be the same, and now we have

$$M = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}$$

which is the QR factorization of M with

$$Q = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Algebraically, the above explanation is encoded as

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}\right) =$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Additional page for solutions 3

Problem 4. *25pts.*

(a) Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}.$$

(b) Draw a sketch showing the vector $\vec{b}$, the image of A, the vector $A\vec{x}^*$, and the vector $\vec{b} - A\vec{x}^*$.

Solution: The least-squares solution is

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

and $\vec{b} - A\vec{x}^* = \vec{0}$, so $\vec{x}^*$ is the exact solution. In particular, $\vec{b}$ is inside the plane $\mathrm{im}(A)$, and it coincides exactly with $A\vec{x}^*$.

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Problem 5. *20pts.*

Consider a 4×4 matrix A with rows $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$. If $\det(A) = 8$, find

$$\det \begin{bmatrix} 6\vec{v_1} + 2\vec{v_4} \\ \vec{v_2} \\ \vec{v_3} \\ 3\vec{v_1} + \vec{v_4} \end{bmatrix}.$$

Solution: The first and the last rows are not linearly independent since we can obtain the last row as 1/2 of the first row, so the determinant is zero.

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