$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

One sol. consistent — No sol. inconsistent — Infinitely sol. consistent

free variable

$$\begin{cases} x + 2y = 1 \\ z = 2 \end{cases} \quad y = t \quad x = 1 - 2t \quad \begin{bmatrix} 1-2t \\ t \\ 2 \end{bmatrix}$$

inconsistent $\to [0 \cdots 0 | 1]$

consistent $\to$ free variable $\to \infty$ sol.

$\to$ no free var. $\to$ 1 sol. (all variables leading)

Rank: number of leading ones in rref(A). This says something about the number of solutions!

There is a relation between the number of equations, the number of variables, and the solutions. If there are no redundant equations, we need n equations to solve for n unknowns, and we need rank n.

Sum of matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$ entry-wise!

Scalar multiples of matrices:

$$4 \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 16 & 20 & 24 \end{bmatrix}$$ entry wise!

Equality of matrices: entry-wise!

Multiplication of matrices: row by column!

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \cdot \underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}} = \begin{bmatrix} 1+6+15 & 2+8+18 \\ 4+15+30 & 8+20+36 \end{bmatrix} = \underset{2\times 2}{\begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}}$$

Dot product of vectors: multiplication of matrices!

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 + 10 + 18 = \underset{1\times 1}{32} = \underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \cdot \underset{3\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}$$

We can interpret a system of linear equations as a matrix equation:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \longleftrightarrow \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{= \begin{bmatrix} x+2y+3z \\ 2x-3y+2z \\ 3x+y-z \end{bmatrix}} = \begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix}$$

Multiplication in terms of ~~rows~~ columns:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \right) =$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot x + \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot y + \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \cdot z$$

Multiplication in terms of rows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 1 & -1 \end{bmatrix} \right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$= \begin{bmatrix} [1 \; 2 \; 3] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ [2 \; -3 \; 2]\begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ [3 \; 1 \; -1]\begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{bmatrix} = \dots$$

Linear combination: a vector is a linear combination of others if it can be written as: $\vec{b} = a_1 \cdot \vec{v}_1 + \dots + a_n \cdot \vec{v}_n$.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$