A <u>function</u> T from X to Y is an assignment of a unique element y of Y to each x of X.

$g(x) = \frac{1}{1-x}$

$h(x) = \sum_{n=0}^{\infty} x^n$

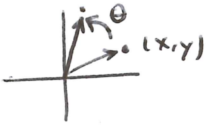
$T: \mathbb{R} \longrightarrow \mathbb{R}^+$, $x \longmapsto e^x$

A function is <u>invertible</u> if it can be undone, namely to each y of Y corresponds exactly one x of X. $f(x) = x^2 : \mathbb{R} \longrightarrow \mathbb{R}$.

$S: \mathbb{R}^+ \longrightarrow \mathbb{R}$, $y \longmapsto \ln(y)$

A <u>linear transformation</u> is a function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ given by multiplying a vector by a matrix. $n \times m$

<u>Example:</u>



We do this component-wise!

$$\begin{bmatrix}1\\0\end{bmatrix} \longmapsto \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix} \longmapsto \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}.$$

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{bmatrix}$$

We always find the associated matrix in this way, by seeing where the vectors: $\mathbb{R}^m$; $\underbrace{\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\ \vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}}_{m}$ are sent, and putting them into columns.

<u>Example:</u> $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$

$$\begin{bmatrix}x\\y\end{bmatrix} \longmapsto \begin{bmatrix}x^2 - y\\ y^2 - x\end{bmatrix}$$

<u>not</u> linear: $\begin{bmatrix}1\\0\end{bmatrix} \longmapsto \begin{bmatrix}1\\-1\end{bmatrix}$, but: $\begin{bmatrix}0\\1\end{bmatrix} \longmapsto \begin{bmatrix}-1\\1\end{bmatrix}$

$$\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x-y\\ y-x\end{bmatrix}$$ which is <u>not</u> what we were given!

<u>Example:</u> inverse of rotation: $\begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix} = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix}x\\y\end{bmatrix} \longmapsto \begin{bmatrix}xc\theta - ys\theta\\ xs\theta + yc\theta\end{bmatrix} \longmapsto \begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{bmatrix}axc\theta - ays\theta\\ + bxs\theta + byc\theta\\ cxc\theta - cys\theta\\ + dxs\theta + dyc\theta\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}xc\theta - ys\theta\\ xs\theta + yc\theta\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

<u>Ex:</u> $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ invertible?

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \longmapsto \begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}x + \begin{bmatrix}4\\5\\6\end{bmatrix}y + \begin{bmatrix}7\\8\\9\end{bmatrix}z$$

$$\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}\equiv\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

A function $\underset{\mathbb{R}^m \to \mathbb{R}^n}{T}$ is a <u>linear transformation</u> exactly when:

(i) $T(x+y) = T(x) + T(y)$.

(ii) $T(a \cdot x) = a \cdot T(x)$.

<u>Ex:</u>

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix} =$$

$$= (x+a)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (y+b)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + (z+c)\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} =$$

$$= \underline{x \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} + \underline{\underline{a \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}} + \underline{y \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} + \underline{\underline{b \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}} + \underline{z \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}} + \underline{\underline{c \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}}}$$

$$= \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

$f(x) = e^x$

$f(x+y) = e^{x+y} = e^x \cdot e^y = f(x) \cdot f(y)$.

$f(a \cdot x) = e^{a \cdot x} = (e^a)^x = (e^x)^a$.