Recall: T linear if $\exists A$ with $T(\vec{v}) = A\cdot\vec{v}$.

T linear if $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$.

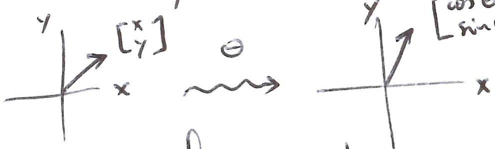
$T(a\cdot\vec{v}) = a\cdot T(\vec{v})$.

To obtain A: $\left[T\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \cdots T\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} \right]$.

Geometric interpretation:

We have already seen that rotations are linear transformations:

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$



Any linear transformation has a geometric interpretation.

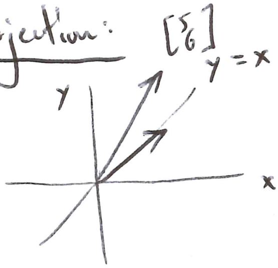
(i) $\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ and $\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$: scaling.

(ii) $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$: projection, projection and scaling.

(iii) $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$: reflection.

(iv) $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$: shears (ugh!)

Projection:



$\begin{bmatrix}1\\1\end{bmatrix} \overset{\text{unit}}{\rightsquigarrow} \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \vec{u}^{\parallel}$

$\begin{bmatrix}5\\6\end{bmatrix}$, write in terms of $\vec{u}^{\parallel}$ and $\vec{u}^{\perp}$

$$\mathrm{proj}_{\begin{bmatrix}1\\1\end{bmatrix}}\begin{bmatrix}5\\6\end{bmatrix} = \left(\begin{bmatrix}5 & 6\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\right)\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}11/2\\11/2\end{bmatrix}$$

How to find $\vec{u}^{\perp}$: $\begin{bmatrix}a\\b\end{bmatrix}$ s.t. $\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = 0$ $\quad \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = 0$

$a + b = 0$

$\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}a\\b\end{bmatrix} = 1$ or normalize!

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightsquigarrow \vec{n}^{\perp} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}. \quad \text{Check! } \vec{n}^{\perp} \cdot \vec{n}^{\parallel} = 0.$$

$$\operatorname{proj}_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \left(\begin{bmatrix} 5 & 6 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

Check! $\begin{bmatrix} 11/2 \\ 11/2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$

Note! projecting on $\vec{n}^{\parallel}$ is subtracting $\vec{x}^{\perp}$, and reflection is subtracting $\vec{x}^{\perp}$ twice!

<u>Reflection:</u>

$$\begin{bmatrix} 5 \\ 6 \end{bmatrix} \rightsquigarrow \underbrace{\underbrace{\begin{bmatrix} 5 \\ 6 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}}_{\substack{\text{proj} \\ \begin{bmatrix} 11/2 \\ 11/2 \end{bmatrix}}} - \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}}_{\begin{bmatrix} 6 \\ 5 \end{bmatrix}}$$

makes sense!