## Dimension:

We saw that $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ are basis of $\mathbb{R}^3$. What other basis of $\mathbb{R}^3$ do we have?

(in particular $\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$, so

<u>no</u> uniqueness for generators). Uniqueness for number!

Any two basis of a vector subspace of $\mathbb{R}^n$ will have the same <u>number</u> of elements. That number is the <u>dimension</u> of the subspace. (sweeping under the rug whether basis exist or not).

<u>Ex</u>: Find a basis of the image and kernel of: $\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix} = A$

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<u>Image</u>: $\text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right)$. Find redundancy:

$$\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix} \xrightarrow[R_2-2R_1]{} \begin{bmatrix}1&4&7\\0&-3&-6\\3&6&9\end{bmatrix} \xrightarrow[R_3-3R_1]{} \begin{bmatrix}1&4&7\\0&-3&-6\\0&-6&-12\end{bmatrix}$$

$$\xrightarrow[\frac{R_2}{-3}]{} \begin{bmatrix}1&4&7\\0&1&2\\0&-6&-12\end{bmatrix} \xrightarrow[R_1-4R_2]{} \begin{bmatrix}1&0&-1\\0&1&2\\0&-6&-12\end{bmatrix}$$

$$\xrightarrow[R_3+6R_2]{} \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}$$

The columns without leading ones are redundant.

$$\text{Image}(A) = \text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right).$$

$$\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Leftrightarrow$$

$$(-1)\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} + (2)\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\2\\0\end{bmatrix}$$

Kernel: $\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$

$$\begin{cases} t = z \text{ free.} \\ x = z \\ y = -2z \end{cases} \qquad \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \quad t \text{ real.}$$

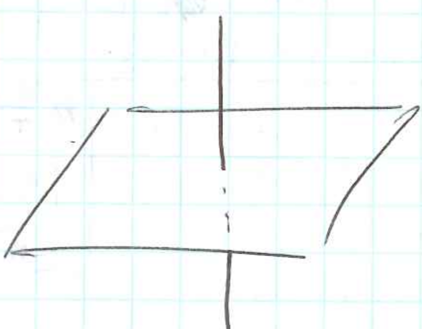
$$\ker(A) = \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \middle| \ t \in \mathbb{R} \right\} = \left\{ t \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$$

Construction: A basis of $\mathrm{Im}(A)$ is given by the column vectors of $A$ corresponding to columns with leading ones.

In particular, dim of image is rank of $A$.

Useful to check: $\dim(\ker(A)) + \dim(\mathrm{im}(A)) = \#$ columns of $A$.

Projections:



Image?

Kernel?

When do vectors $\vec{v}_1, \ldots, \vec{v}_m$ form a basis of $\mathbb{R}^n$?

The matrix:

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$ needs to be invertible.

How many linearly independent vectors can we find in $\mathbb{R}^n$?
How many vectors do we need to span $\mathbb{R}^n$?