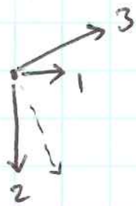
Gram-Schmidt and QR factorization.

Gram-Schmidt transforms any basis into an orthonormal basis.

choose one direction, make it length 1.

project the second, keep orthogonal component.



make second of length 1.

project the third, keep orthogonal component.

make third of length 1.

Example: $\mathbb{R}^3 = \text{Span}\left(\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{\vec{v}_3}\right)$, find orthonormal basis.

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{\|\vec{v}_1\|}\cdot\vec{v}_1 \qquad r_{11} = \|\vec{v}_1\|.$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - \left(\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix}1\\-1\\2\end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\cdot\vec{v}_2^{\perp} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}.$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \mathrm{proj}_{(u_1,u_2)}\vec{v}_3 = \vec{v}_3 - \left(\begin{bmatrix}0\\1\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$-\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$-\frac{1}{6}\cdot 1\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}-\frac{1}{6}\\ \frac{1}{2}+\frac{1}{6}\\ 1-\frac{2}{6}\end{bmatrix} = \begin{bmatrix}-4/6\\4/6\\4/6\end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|}\cdot\vec{v}_3^{\perp} = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}| & | & |\\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3\\ | & | & |\end{bmatrix}\cdot R =$$

$$= \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}\cdot R$$

$$R = \begin{bmatrix}\|\vec{v}_1\| & \vec{u}_1\cdot\vec{v}_2 & \vec{u}_1\cdot\vec{v}_3\\ 0 & \|\vec{v}_2^{\perp}\| & \vec{u}_2\cdot\vec{v}_3\\ 0 & 0 & \|\vec{v}_3^{\perp}\|\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix},\ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

$$\begin{bmatrix}A\end{bmatrix} = \begin{bmatrix}u\end{bmatrix}\cdot R$$

$$\begin{bmatrix}\overbrace{Q\cdot cR_1 \quad Q\cdot cR_2 \quad Q\cdot cR_3}^{Q}\end{bmatrix}$$