Properties of the determinant.

Recall: The determinant is a function assigning a number to a square matrix.

For example: $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \longrightarrow -8$.

How to compute the determinant: we expand.

The determinant of A coincides with the determinant of A^T.

Since the determinant was linear in the columns,

$$\det \begin{bmatrix} 1+1 & 2 \\ 1+1 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

alternating in the columns,

$$\det \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} = 0$$

and the determinant of the identity is 1; $\quad \det(A) = \det(A^T)$

Theorem: then the determinant is linear in the rows, alternating in the rows, and the determinant of the identity is 1.

Question: How does the determinant change when we do elementary row operations?

Theorem: 1) If matrix B is obtained from matrix A by dividing a row by k a real number, then $\det(B) = \frac{1}{k} \cdot \det(A)$

$$A \xrightarrow[\frac{R}{k}]{} B \qquad \det(B) = \frac{1}{k}\cdot\det(A).$$

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \qquad \det(A) = 1\cdot 3 - 2\cdot 5 = -7$$

$$\Big\downarrow \frac{R_1}{8}$$

$$B = \begin{bmatrix} 1/8 & 5/8 \\ 2 & 3 \end{bmatrix} \qquad \det(B) = \frac{1}{8}\cdot 3 - 2\cdot\frac{5}{8} = \frac{1\cdot 3 - 2\cdot 5}{8} = \frac{-7}{8} = \frac{\det(A)}{8}$$

2) If matrix B is obtained from matrix A by swapping two rows, then $\det(B) = -\det(A)$.

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \qquad \det(B) = 2\cdot 5 - 1\cdot 3 = 7 = -\det(A).$$

3) If matrix B is obtained from matrix A by adding a row to another row then $\det(B) = \det(A)$.

$$R_1 + R_2 \qquad B = \begin{bmatrix} 3 & 8 \\ 2 & 3 \end{bmatrix} \qquad \det(B) = \det\begin{bmatrix} 1+2 & 5+3 \\ 2 & 3 \end{bmatrix} = \overbrace{\det\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}}^{A} + \det\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \det(A) + 0 = \det(A) = -7.$$

<u>Theorem:</u> Given A, compute $\mathrm{rref}(A)$. If $\mathrm{rref}(A)$ is <u>not</u> the identity, then A is <u>not</u> invertible, so $\det(A) = 0$. If $\mathrm{rref}(A)$ is the

(A is invertible if and only if $\det(A) \neq 0$)

identity, count how many times rows were swapped and when and by how much were rows divided by scalars. Say rows were swapped s times. Say we divided by $k_1, k_2, \ldots, k_r$. Then:

$$\det(A) = (-1)^s \cdot k_1 \cdot k_2 \cdots k_r.$$

<u>Example:</u> Compute det of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$. We first row-reduce.

$$\underset{B}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{-4}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow[R_3 + 3R_2]{R_1 - 2R_2}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - 2R_1]{R_1 + R_3} \underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} = (-1)^0 \cdot \underline{(-4)} \cdot \underline{\underline{2}} = 1 \cdot (-4) \cdot 2 = -8.$$

$$\det(B) = \boxed{} \cdot \det(A) = \boxed{}$$

<u>Theorem:</u> $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

<u>Exercise:</u> Compute $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}^{-1}$ and check it has det. $\dfrac{1}{-8}$.

<u>Theorem:</u> $\det(A \cdot B) = \det(A) \cdot \det(B)$.

The det of a multiplication is the multiplication of the dets.

⚠ It is <u>not</u> true that $\det(A+B) = \det(A) + \det(B)$.

<u>Exercise:</u> Find $\underset{2\times 2}{A}$ such that $\det(A+A) \neq \det(A) + \det(A)$.

$$\underbrace{\det(A+A)}_{\det(2\cdot A)} \neq \underbrace{\det(A)+\det(A)}_{2\cdot \det(A)}.$$

$$\underbrace{\det(2\cdot A)}_{\overset{\shortparallel}{4\cdot\det(A)}} \neq 2\cdot\det(A).$$