Geometric interpretation of the determinant.

We'll use Gram-Schmidt and QR factorization.

Recall: $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$ invertible.

why? So that $\det(A) \neq 0$.

$A = Q \cdot R$ with

$Q = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}$ orthogonal matrix and (because square, only orthonormal columns in general)

$$R = \begin{array}{cc} \begin{bmatrix} \|\vec{v}_1\| & u_1 \cdot v_2 & u_1 \cdot v_3 \\ 0 & \|\vec{v}_2^{\perp}\| & u_2 \cdot v_3 \\ 0 & 0 & \|\vec{v}_3^{\perp}\| \end{bmatrix} & \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \\ \begin{array}{ccc} v_1 & \quad v_2 & \quad v_3 \end{array} & \end{array}$$

upper triangular matrix.

Then: $\det(A) = \det(QR) = \det(Q) \cdot \det(R) = \pm \det(R) =$

$= \pm \|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\| \cdot \|\vec{v}_3^{\perp}\|.$

Thm: The determinant of an orthogonal matrix is ± 1.

Thm: The determinant of a triangular matrix is the multiplication of the diagonal elements.

Thm: The determinant of $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ is

$\|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\| \cdots \|\vec{v}_n^{\perp}\|.$

perpendicular to everything before.

So, the determinant of $\vec{v}_1, \dots, \vec{v}_n$ is the volume of the parallelepiped determined by $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$.

Cramer's rule:

It's a closed formula for the solutions of a linear system.

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad A\vec{x} = \vec{b}.$$

The components of the solution $\vec{x}$ are:

$$x = \frac{\det \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 0 \end{bmatrix}}{\det(A)} = \frac{-2}{-2} = 1$$

$$y = \frac{\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}}{\det(A)} = \frac{-4}{-2} = 2$$

$$z = \frac{\det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix}}{\det(A)} = \frac{8}{-2} = -4$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \downarrow \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -3 & -1 & 3 \end{bmatrix} \downarrow \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -3 & 1 & 3 \end{bmatrix} \downarrow \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & -1 & -3 \end{bmatrix}\frac{1}{2}$$

Thm: Solution of $A\vec{x} = \vec{b}$ are:

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \qquad x_i = \frac{\det \begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{b} & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix}}{\det(A)}.$$

This is conceptually very useful because it gives a formula for the inverse!

Thm: $$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}. \qquad \operatorname{adj}(A)_{ij} = (-1)^{i+j} \det(A_{ji}).$$

$A \to A^T \to$ compute determinants $\to$ add signs $\to$ divide $\det(A)$