Diagonalization:

Real motivation: classification of linear transformations.

For us: useful computational tool.

Question: Is the matrix $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ similar to a diagonal matrix? Namely, is there a basis $\mathcal{B}$ such that the linear transformation given by A in the basis $\mathcal{B}$ has a diagonal associated matrix? Answer: $D=\begin{bmatrix}2&0\\0&0\end{bmatrix}$ $S=\frac{1}{2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

Recall: A change of basis does not change the linear transformation. Similar matrices give the same linear transformation, just in different basis.

Def: $T(\vec{x})=A\vec{x}$ is *diagonalizable* iff $\exists$ basis $\mathcal{B}$ with matrix D of T with respect to $\mathcal{B}$ is diagonal.
Equivalently, A is diagonalizable iff A is similar to a diagonal matrix D.
That is, $\exists$ invertible matrix S such that $S^{-1}AS=D$ diagonal.

To *diagonalize* a matrix A means to find an invertible matrix S such that $S^{-1}AS$ is diagonal.

For this, we use eigenvectors and eigenvalues. These are *preferential directions* of the matrix:

Def: $T(\vec{x})=A\vec{x}$. A vector in $\mathbb{R}^n$ is an eigenvector if:

(i) it is not zero.
(ii) $A\vec{v}=\lambda\vec{v}$ for some scalar $\lambda\in\mathbb{R}$.

We call λ the *eigenvalue* associated to $\vec{v}$.

A basis $\mathcal{B}$ of $\mathbb{R}^n$ where every vector is an eigenvector is called an *eigenbasis* of $\mathbb{R}^n$ for A.

Thm: A diagonalizable if and only if there is an eigenbasis $\vec{v}_1, \dots, \vec{v}_n$ of $\mathbb{R}^n$ for A.

(i) Then: $S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$, $S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

(ii) Conversely if $S^{-1}AS$ is diagonal then the columns of S form an eigenbasis of $\mathbb{R}^n$ for A.

Q: Is $\vec{v} = 0$ an eigenvector?
Is $\lambda = 0$ an eigenvalue?

Thm: rotations are not diag.

Thm: projections and reflections are diagonalizable.

Ex: (i) Find matrix of projection onto plane $x+y+z=0$.

$\mathrm{im}(P) = \mathrm{span}\Big(\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{A}\Big)$. $P = A(A^TA)^{-1}A^T$.

Now $P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} =$

$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} =$

$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} =$

$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

(ii) Find preferential directions: everything in the plane, and everything perpendicular. Say $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(iii) Diagonalize: $S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ $S^{-1}PS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.