Finding the eigenvectors of a matrix:

$\rightarrow$ $A\vec{v} = \lambda\vec{v}$ we know λ, we solve for $\vec{v}$.

So: $(A - \lambda \cdot I_n)\vec{v} = 0$ and $\vec{v}$ is in $\ker(A - \lambda \cdot I_n)$.

Def: eigenspace E_λ.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ have $\lambda = 0$, $\lambda = 2$.

$$E_0 = \ker\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

leading 1

free

$x = -y$ y free: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ spans.

$$E_1 = \ker\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right) \quad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x = y \quad y \text{ free}: \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ spans.}$$

check work! $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Ex: $A = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ have $\lambda = 0$, $\lambda = 1$.

$$E_0 = \ker\left(\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\right) \quad \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \xrightarrow[R_1/2,\ R_3 + R_1]{R_2 + R_1}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 3/2 & -3/2 & 0 \\ 0 & -3/2 & 3/2 & 0 \end{array}\right] \xrightarrow[R_3 \cdot 2,\ R_3 + R_2]{R_2 \cdot 2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

leading 1 leading 1 free

$x = z$
$y = z$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ spans.
z free

$$E_1 = \ker \frac{1}{3}\begin{bmatrix} 2-3 & -1 & -1 \\ -1 & 2-3 & -1 \\ -1 & -1 & 2-3 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x = -y - z \\ y \text{ free} \\ z \text{ free.} \end{array}$$

$$E_1 = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

Def: geometric multiplicity of λ: $\dim(E_\lambda)$.

Thm: 1) Vectors in different E_λ are linearly independent.

2) A diagonalizable iff concatenating bases of E_λ's gives an eigenbasis.

3) A diag $\Leftrightarrow$ geometric multiplicities add up to n.

4) A diag $\Leftrightarrow$ algmult(λ) = geomult(λ) $\forall \lambda$.

Life in easy mode: $\underset{n\times n}{A}$ has n distinct eigenvalues. Then A diag.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has algmult$(1) = 2$

$f_A(x) = (x-1)^2$ $\lambda = 1$, geomult$(1) = 1$.

$\ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So A not diag.

How to find eigenbasis:

1) Find λ.
2) Find E_λ.
3) Concatenate basis of E_λ.
4) make sure algmult(λ) = geomult(λ) $\forall \lambda$.