Symmetric matrices:

We have answered: when is A square diagonalizable? By: when A has eigenvalues λ such that $\text{algmult}(\lambda) = \text{geomult}(\lambda)$.

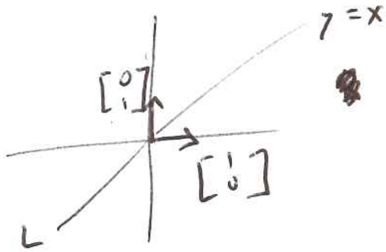
$$S^{-1} A S = D$$

Now, S is a change of basis, it would be great if $S^{-1} = S^T$, computations would be easy.

Question: When is A square such that there is S orthogonal with $S^T A S = D$ diagonal?

Def: Let A be such that $\exists$ S invertible with $S^T A S = D$ diagonal. We say that A is orthogonally diagonalizable.

Example:
1) Diagonal matrices.
2) Projections: $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
3) Reflections:



$$\text{refl}_L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad S^{-1} \cdot \text{proj}_L \cdot S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

$$S^{-1} \cdot \text{refl}_L \cdot S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Q: How many real eigenvalues does a 3×3 matrix have? At least one!

Question: Let A be orthogonally diagonalizable: $S^T A S = D$.

What is the relation between A and A^T?

$$A = SDS^T \quad \text{so} \quad A^T = (SDS^T)^T = (S^T)^T D^T S^T =$$
$$= SDS^T = A \quad \text{is symmetric!}$$

Question: If A is symmetric, is A diagonalizable?

Yes! That's called the Spectral Theorem.

Thm: A is orthogonally diagonalizable if and only if $A^T = A$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $\quad \lambda_+ = 2 + \sqrt{5}$, $\lambda_- = 2 - \sqrt{5}$ $\quad \vec{v}_+ = \begin{bmatrix} \dfrac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}} \\ \sqrt{\dfrac{5+\sqrt{5}}{10}} \end{bmatrix}$.

$\vec{v}_- = \begin{bmatrix} \dfrac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}} \\ \sqrt{\dfrac{5-\sqrt{5}}{10}} \end{bmatrix}$.

Method for orthogonal diag:

1) Eigenvalues. 2) Eigenspaces with orthonormal basis. 3) Put together.

Recall: Eigenvectors with distinct eigenvalues are linearly independent.

Thm: Eigenvectors of a symmetric matrix with distinct eigenvalues are perpendicular.

Recall: A matrix has at most n real ~~eigenvectors~~ eigenvalues counted with multiplicity.

(A matrix has exactly n complex ~~eigenvectors~~ eigenvalues counted with multiplicity.)

Thm: A symmetric matrix has exactly n real eigenvalues counted with multiplicity.