Proposition: Let $A \in M_n(\mathbb{Z})$ invertible. Then $A^{-1} \in M_n(\mathbb{Z})$ if and only if $\det(A) = \pm 1$.

Proof: $\Rightarrow$) We have $A, A^{-1} \in M_n(\mathbb{Z})$, so $\det(A), \det(A^{-1}) \in \mathbb{Z}$. Also:

$$\frac{1}{\det(A)} = \det(A^{-1}) \in \mathbb{Z} \text{ so } \det(A) \text{ is an invertible integer, namely } \det(A) = \pm 1.$$

$\Leftarrow$) We have $A \in M_n(\mathbb{Z})$ and $\det(A) = \pm 1$, so $\det(A_{ij}) \in \mathbb{Z}$ for all $1 \leq i,j \leq n$

and $\frac{1}{\det(A)} = \pm 1$. Also for $1 \leq i,j \leq n$ we have:

$$(A^{-1})_{ij} = \frac{1}{\det(A)} \cdot (\operatorname{adj}(A))_{ij} = \pm (\operatorname{adj}(A))_{ij} = \pm \cdot (-1)^{i+j} \cdot \det(A_{ji}) \in \mathbb{Z}$$

whence $A^{-1} \in M_n(\mathbb{Z})$. $\square$.