Math 33A
Linear Algebra and Applications

Discussion 2

Problem 1($\star$).

Show that if T is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$T\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 T(\vec{e_1}) + \cdots + x_m T(\vec{e_m}),$$

where $\vec{e_1}, \ldots, \vec{e_m}$ are the standard vectors in $\mathbb{R}^m$.

Solution: We can rewrite

$$T\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_m \end{bmatrix}\right) = T\left(x_1\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_m\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right) =$$

$$= T(x_1\vec{e_1} + \cdots + x_m\vec{e_m}) = T(x_1\vec{e_1}) + \cdots + T(x_m\vec{e_m}) = x_1 T(\vec{e_1}) + \cdots + x_m T(\vec{e_m}),$$

where the last two equalities are using that T is linear.

Problem 2.

Describe all linear transformations from $\mathbb{R}$ to $\mathbb{R}$. What do their graphs look like?

Solution: The linear transformations are of the form $[y] = [a][x]$ for some real number a. They encode the equation $y = ax$, which are lines through the origin.

Problem 3.

Describe all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$. What do their graphs look like?

Solution: The linear transformations are of the form $[z] = \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ for some real numbers a, b. They encode the equation $z = ax + by$, which is a plane through the origin.

Problem 4.

Consider two linear transformations $\vec{y} = T(\vec{x})$ and $\vec{z} = L(\vec{y})$, where T goes from $\mathbb{R}^m$ to $\mathbb{R}^p$ and L goes from $\mathbb{R}^p$ to $\mathbb{R}^n$. Is the transformation $\vec{z} = L(T(\vec{x}))$ linear as well?

Solution: Yes. We can check that for vectors $\vec{x}, \vec{x_1}, \vec{x_2}$ and constant k we have

$$\begin{aligned} L(T(\vec{x_1}+\vec{x_2})) =& L(T(\vec{x_1})+T(\vec{x_2})) = L(T(\vec{x_1})+L(T(\vec{x_2})) \\ L(T(k\vec{x})) =& L(kT(\vec{x})) = kL(T(\vec{x})) \end{aligned}$$

so the transformation LT is linear.

Problem 5.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Find the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$.

Solution: We apply T to the vectors $\vec{e_1}$ and $\vec{e_2}$, and then T will have matrix $[T(\vec{e_1})\, T(\vec{e_2})]$. Since

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = B\left(A\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = B\left(\begin{bmatrix} a \\ c \end{bmatrix}\right) = \begin{bmatrix} pa+qc \\ ra+sc \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = B\left(A\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = B\left(\begin{bmatrix} b \\ d \end{bmatrix}\right) = \begin{bmatrix} pb+qd \\ rb+sd \end{bmatrix}$$

the associated matrix is

$$\begin{bmatrix} pa+qc & pb+qd \\ ra+sc & rb+sd \end{bmatrix}.$$