Math 33A
Linear Algebra and Applications

Discussion 3

Problem 1.

Show that if a square matrix A has two equal columns, then A is not invertible.

Solution: Applying an elementary row operation to a matrix with two equal columns will result in a matrix with two equal columns. Say A is an $n \times n$ matrix, then $\text{rref}(A)$ has two equal columns, so it is not I_n. To find the inverse of A, we would compute $\text{rref}([A|I_n])$, but since $\text{rref}(A) \neq I_n$ then $\text{rref}([A|I_n]) \neq [I_n|B]$, so A is not invertible.

Problem 2($\star$).

Which of the following linear transformations T from $\mathbb{R}^3$ to $\mathbb{R}^3$ are invertible? Find the inverse if it exists.

(a) Reflection about a plane.

(b) Orthogonal projection onto a plane.

(c) Scaling by a real factor (namely, fix a real number r and consider $T(\vec{v}) = r\vec{v}$, for all vectors $\vec{v}$).

(d) Rotation about an axis.

Solution:

1. Invertible, this transformation is its own inverse.
2. Not invertible, if $\vec{v}$ is a vector perpendicular to the plane, then all the vectors $k\vec{v}$ for k a real number are sent to the same vector.
3. Invertible, the inverse is scaling by $1/r$.
4. Invertible, the inverse is rotating about the same axis in the opposite direction.

Problem 3.

A square matrix is called a permutation matrix if it contains a 1 exactly once in each row and in each column, with all other entries being 0. Give an example of two different 3×3 permutation matrices.

Solution: Two different permutation matrices are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Problem 4.

Are permutation matrices invertible? If so, is the inverse a permutation matrix as well?

Solution: Yes. Yes. Let A be an $n \times n$ permutation matrix, then to go from $\text{rref}([A|I_n])$ to $\text{rref}([I_n|B])$ we only need to permute rows, so B will have a 1 exactly once in each row and column.

Problem 5.

Consider two invertible $n \times n$ matrices A and B. Is the linear transformation $\vec{y} = A(B(\vec{x}))$ invertible? If so, what is the inverse?

Solution: Yes, the inverse is $\vec{x} = B^{-1}(A^{-1}(\vec{y}))$.

Problem 6.

Are the columns of an invertible matrix linearly independent?

Solution: Yes. If A is invertible then $\ker(A) = \{0\}$ so its columns are linearly independent.

Problem 7.

Consider linearly independent vectors $\vec{v_1}, \ldots, \vec{v_m}$ in $\mathbb{R}^n$, and let A be an invertible $m \times m$ matrix. Are the columns of the following matrix linearly independent?

$$\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix} A$$

Solution: Yes. If A is invertible then $\ker(A) = \{\vec{0}\}$ so the columns of A are linearly independent. Also, since all the vectors $\vec{v_1}, \ldots, \vec{v_m}$ are linearly independent, then $\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix}$ has kernel $\{\vec{0}\}$. If $\vec{x}$ is in the kernel of $\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix} A$ then $A\vec{x}$ is in the kernel of $\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix}$, so $A\vec{x} = \vec{0}$. Thus $\vec{x}$ is in the kernel of A, so $\vec{x} = \vec{0}$.

This means that the matrix $\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix} A$ has kernel $\{\vec{0}\}$, so all its columns are linearly independent.