Math 33A
Linear Algebra and Applications

Discussion 5

Problem 1.

Here is an infinite-dimensional version of Euclidean space: In the space of all infinite sequences, consider the subspace ℓ_2 of square-summable sequences (namely, those sequences $(x_1, x_2, \dots)$ for which the infinite series $x_1^2 + x_2^2 + \cdots$ converges). For x and y in ℓ_2, we define

$$||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots} \quad \text{and} \quad \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots .$$

A preliminary question is, why do $||\vec{x}||$ and $\vec{x} \cdot \vec{y}$ make sense, that is, why are they finite real numbers?

(a) Check that $\vec{x} = (1, 1/2, 1/4, 1/8, 1/16, \dots)$ is in ℓ_2, and find $||\vec{x}||$. Recall the formula for the geometric series: $1 + a + a^2 + a^3 + \cdots = 1/(1-a)$ if $-1 < a < 1$.

(b) Find the angle between $(1, 0, 0, 0, \dots)$ and $(1, 1/2, 1/4, 1/8, \dots)$.

(c) Give an example of a sequence $(x_1, x_2, \dots)$ that converges to 0 ($\lim_{n\to\infty} x_n = 0$) but does not belong to ℓ_2.

(d) Let L be the subspace of ℓ_2 spanned by $(1, 1/2, 1/4, 1/8, \dots)$. Find the orthogonal projection of $(1, 0, 0, 0, \dots)$ onto L.

The Hilbert space ℓ_2 was initially used mostly in physics: Werner Heisenberg's formulation of quantum mechanics is in terms of ℓ_2. Today, this space is used in many other applications, including economics. See, for example, the work of the economist Andreu Mas-Colell of the University of Barcelona.

Solution:

(a) Using the formula for the geometric series $||\vec{x}||^2 = 4/3$ so $||\vec{x}|| = 2/\sqrt{3}$.

(b) Set $\vec{x} = (1, 0, 0, 0, \dots)$ and $\vec{y} = (1, 1/2, 1/4, 1/8, \dots)$, then

$$\theta = \arccos\left(\frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| \cdot ||\vec{y}||}\right) = \arccos\left(\frac{1}{2/\sqrt{3}}\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

(c) Consider $\vec{x} = (1, 1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{4}, \dots)$, then

$$||\vec{x}||^2 = \sqrt{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots} = \sqrt{\sum_{n=1}^{\infty} \frac{1}{n}}$$

which diverges since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(d) Let $\vec{x} = (1, 0, 0, 0, \dots)$ and $\vec{y} = (1, 1/2, 1/4, 1/8, \dots)$, we want the orthogonal projection of $\vec{x}$ onto $L = \text{span}(\vec{y})$. For this, we first find a vector of length one in the direction of $\vec{y}$, namely

$$\vec{u} = \frac{\vec{y}}{||\vec{y}||} = \frac{\sqrt{3}}{2}\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right)$$

and now we compute

$$\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u} = \left(\frac{\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2}\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right) = \left(\frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \dots\right).$$

Problem 2.

Give an algebraic proof for the triangle inequality

$$||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||.$$

Draw a sketch.

Solution: Note that

$$\begin{aligned} ||\vec{v} + \vec{w}||^2 &= (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{w} = \\ &= ||\vec{v}||^2 + 2(\vec{v} \cdot \vec{w}) + ||\vec{w}||^2 \leq ||\vec{v}||^2 + 2(||\vec{v}|| \cdot ||\vec{w}||) + ||\vec{w}||^2 = (||\vec{v}|| + ||\vec{w}||)^2 \end{aligned}$$

where we have used the Cauchy-Schwarz inequality. Thus $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$.

Problem 3($\star$).

(a) Consider a vector $\vec{v}$ in $\mathbb{R}^n$, and a scalar k. Show that $||k\vec{v}|| = |k|||\vec{v}||$.

(b) Show that if $\vec{v}$ is a nonzero vector in $\mathbb{R}^n$, then $\vec{u} = \frac{\vec{v}}{||\vec{v}||}$ is a unit vector.

Solution:

(a) Note that

$$||k\vec{v}||^2 = (k\vec{v}) \cdot (k\vec{v}) = k^2(\vec{v} \cdot \vec{v}) = k^2||\vec{v}||^2$$

and thus taking square roots $||k\vec{v}|| = |k|||\vec{v}||$ since $|k| = \sqrt{k^2}$.

(b) We compute

$$||\vec{u}|| = \left|\left|\frac{\vec{v}}{||\vec{v}||}\right|\right| = \left|\left|\frac{1}{||\vec{v}||}\vec{v}\right|\right| = \frac{1}{||\vec{v}||}||\vec{v}|| = 1$$

using what we just proved.

Problem 4.

Can you find a line L in $\mathbb{R}^n$ and a vector $\vec{x}$ in $\mathbb{R}^n$ such that $\vec{x} \cdot \mathrm{proj}_L \vec{x}$ is negative? Explain, arguing algebraically.

Solution: No. Let $\vec{x} = \vec{x}^{||} + \vec{x}^{\perp}$ be the decomposition of $\vec{x}$ into the components parallel and perpendicular to L. In particular $\vec{x}^{||} = \mathrm{proj}_L \vec{x}$ and $\vec{x}^{\perp} \cdot \vec{x}^{||} = 0$. Now

$$\vec{x} \cdot \mathrm{proj}_L \vec{x} = (\vec{x}^{||} + \vec{x}^{\perp}) \cdot \vec{x}^{||} = \vec{x}^{||} \cdot \vec{x}^{||} + \vec{x}^{\perp} \cdot \vec{x}^{||} = ||\vec{x}^{||}||^2 \geq 0.$$