Math 33A
Linear Algebra and Applications

Discussion 7

Problem 1.

The following determinant was introduced by Alexandre-Theophile Vandermonde. Consider distinct real numbers $a_0, \dots, a_n$, we define the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{bmatrix}.$$

Vandermonde showed that $\det(A) = \prod_{i>j} (a_i - a_j)$, the product of all differences $a_i - a_j$, where i exceeds j.

(a) Verify this formula in the case of $n = 1$.

(b) Suppose the Vandermonde formula holds for $n-1$. You are asked to demonstrate it for n. Consider the function

$$f(t) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^n & a_1^n & \cdots & a_{n-1}^n & t^n \end{bmatrix}.$$

Explain why $f(t)$ is a polynomial of n-th degree. Find the coefficient k of t^n using Vandermonde's formula for $a_0, \dots, a_{n-1}$. Explain why $f(a_0) = f(a_1) = \cdots = f(a_{n-1}) = 0$. Conclude that $f(t) = k(t - a_0)(t - a_1) \cdots (t - a_{n-1})$ for the scalar k you found above. Substitute $t = a_n$ to demonstrate Vandermonde's formula.

Problem 2(⋆).

Find

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix}$$

using Vandermonde's formula and using the usual definition of determinant.

Problem 3.

For n distinct scalars $a_1, \dots, a_n$, find

$$\det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{bmatrix}.$$