Math 33A
Linear Algebra and Applications

Discussion 7

Problem 1.

The following determinant was introduced by Alexandre-Theophile Vandermonde. Consider distinct real numbers $a_0, \ldots, a_n$, we define the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{bmatrix}.$$

Vandermonde showed that $\det(A) = \prod_{i>j} (a_i - a_j)$, the product of all differences $a_i - a_j$, where i exceeds j.

(a) Verify this formula in the case of $n = 1$.

(b) Suppose the Vandermonde formula holds for $n-1$. You are asked to demonstrate it for n. Consider the function

$$f(t) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^n & a_1^n & \cdots & a_{n-1}^n & t^n \end{bmatrix}.$$

Explain why $f(t)$ is a polynomial of n-th degree. Find the coefficient k of t^n using Vandermonde's formula for $a_0, \ldots, a_{n-1}$. Explain why $f(a_0) = f(a_1) = \cdots = f(a_{n-1}) = 0$. Conclude that $f(t) = k(t - a_0)(t - a_1) \cdots (t - a_{n-1})$ for the scalar k you found above. Substitute $t = a_n$ to demonstrate Vandermonde's formula.

Solution:

(a) For $n = 1$ we have

$$A = \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix} \quad \text{so} \quad \det(A) = a_1 - a_0$$

and the formula holds.

(b) Suppose that the formula holds for $n-1$, let

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & & \vdots & \\ a_0^n & a_1^n & \cdots & a_{n-1}^n & t^n \end{bmatrix}$$

and expand down the rightmost column. This yields

$$
\begin{aligned}
f(t) &= \det(B) = \\
&= \sum_{i=0}^{n-1}(-1)^{i+1+n+1}t^i \det(B_{i+1,n+1}) + (-1)^{n+1+n+1}t^n \det(B_{n+1,n+1}) = \\
&= \sum_{i=0}^{n-1}(-1)^{i+n}t^i \det(B_{i,n}) + t^n \prod_{n-1\geq i>j}(a_i - a_j)
\end{aligned}
$$

where $\det(B_{n+1,n+1}) = \prod_{n-1\geq i>j}(a_i - a_j)$ is the Vandermonde formula for $n-1$. Moreover $f(a_0) = \cdots = f(a_{n-1}) = 0$ since in each case we are computing the determinant of a matrix that has two identical columns. Hence $f(t)$ is a polynomial of degree n that has the n real numbers $a_0, \ldots, a_{n-1}$ as roots, and the coefficient of t^n is $\prod_{n-1\geq i>j}(a_i - a_j)$, so

$$f(t) = \left(\prod_{n-1\geq i>j}(a_i - a_j)\right)(t - a_0)\cdots(t - a_{n-1}).$$

Thus

$$\det(A) = f(a_n) = \left(\prod_{n-1\geq i>j}(a_i - a_j)\right)(a_n - a_0)\cdots(a_n - a_{n-1}) = \prod_{n\geq i>j}(a_i - a_j)$$

as desired.

Problem 2($\star$).

Find

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix}$$

using Vandermonde's formula and using the usual definition of determinant.

Solution: We have $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $a_4 = 5$, and

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix} = \prod_{4\geq i>j}(a_i - a_j) = 288.$$

Problem 3.

For n distinct scalars $a_1, \ldots, a_n$, find

$$\det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{bmatrix}.$$

Solution: Factoring out one a_i from the i-th column, consecutively, we find that

$$\begin{aligned}
\det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{bmatrix} &= a_1 \det \begin{bmatrix} 1 & a_2 & \cdots & a_n \\ a_1 & a_2^2 & \cdots & a_n^2 \\ a_1^2 & a_2^3 & \cdots & a_n^3 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^n & \cdots & a_n^n \end{bmatrix} = \\
&= a_1 a_2 \det \begin{bmatrix} 1 & 1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n^2 \\ a_1^2 & a_2^2 & \cdots & a_n^3 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^n \end{bmatrix} = \cdots = \\
&= (a_1 \cdots a_n) \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix} = \\
&= \left(\prod_{i=1}^{n} a_i \right) \left(\prod_{n \geq i > j} (a_i - a_j) \right)
\end{aligned}$$

where we have used the Vandermonde formula for the last determinant.