Math 33A
Linear Algebra and Applications

Discussion 9

Problem 1.

Consider the $n \times n$ matrix

$$J_n(k) = \begin{bmatrix} k & 1 & 0 & \cdots & 0 & 0 \\ 0 & k & 1 & \cdots & 0 & 0 \\ 0 & 0 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k & 1 \\ 0 & 0 & 0 & \cdots & 0 & k \end{bmatrix}$$

(with all k's on the diagonal and 1's directly above), where k is an arbitrary constant. Find the eigenvalue(s) of $J_n(k)$, and determine their algebraic and geometric multiplicities.

Problem 2($\star$).

Are the following matrices similar?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 3.

Consider a nonzero 3×3 matrix A such that $A^2 = 0$.

(a) Show that the image of A is a subspace of the kernel of A.

(b) Find the dimensions of the image and kernel of A.

(c) Pick a nonzero vector v_1 in the image of A, and write $\vec{v_1} = A\vec{v_2}$ for some $\vec{v_2}$ in $\mathbb{R}^3$. Let $\vec{v_3}$ be a vector in the kernel of A that fails to be a scalar multiple of $\vec{v_1}$. Show that $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a basis of $\mathbb{R}^3$.

(d) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to basis $\mathfrak{B}$.

Problem 4.

If A and B are two nonzero 3×3 matrices such that $A^2 = B^2 = 0$, is A necessarily similar to B?

Problem 5.

For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix},$$

find an invertible matrix S such that

$$S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 6.

Consider an $n \times n$ matrix A such that $A^2 = 0$, with $\operatorname{rank}(A) = r$ (above we have seen the case $n = 3$ and $r = 1$). Show that A is similar to the block matrix

$$B = \begin{bmatrix} J & 0 & \cdots & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & J & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \quad \text{where} \quad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Matrix B has r blocks of the form J along the diagonal, with all other entries being 0. To show this, proceed as in the case above: Pick a basis $\vec{v_1}, \ldots, \vec{v_r}$ of the image of A, write $\vec{v_i} = A\vec{w_i}$ for $i = 1, \ldots, r$, and expand $\vec{v_1}, \ldots, \vec{v_r}$ to a basis $\vec{v_1}, \ldots, \vec{v_r}, \vec{u_1}, \ldots, \vec{u_m}$ of the kernel of A. Show that $\vec{v_1}, \vec{w_1}, \vec{v_2}, \vec{w_2}, \ldots, \vec{v_r}, \vec{w_r}, \vec{u_1}, \ldots, \vec{u_m}$ is a basis of $\mathbb{R}^n$, and show that B is the matrix of $T(\vec{x}) = A\vec{x}$ with respect to this basis.