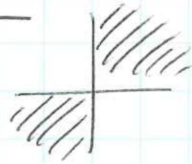
Subspaces of $\mathbb{R}^n$, bases, linear independence:

Def: vector subspace of $\mathbb{R}^n$: (i) $\vec{0}$ (ii) + (iii)

Example: 1) Vector subspaces of $\mathbb{R}$.
2) — — $\mathbb{R}^2$.
3) — — $\mathbb{R}^3$.
n) — — $\mathbb{R}^n$.

Non-example: $\mathbb{R}^2$



Def: The vectors $\vec{v}_1, \dots, \vec{v}_n$ are <u>linearly independent</u>

if there are no scalars $a_1, \dots, a_n$ in $\mathbb{R}$ such that:
non-zero

$$a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{0}.$$

The vectors $\vec{v}_1, \dots, \vec{v}_n$ are <u>linearly dependent</u> if

they are not linearly independent: there exist scalars
non-zero

$a_1, \dots, a_n$ in $\mathbb{R}$ s.t.: $a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{0}$.

Example: Vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ are linearly independent.

$$\vec{0} = x\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} + y\cdot\begin{bmatrix}1\\0\\1\end{bmatrix} + z\cdot\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&0&0\\1&0&1&0\\0&1&1&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&0&0\\0&-1&1&0\\0&1&1&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&0&0\\0&1&1&0\\0&-1&1&0\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&1&0\\0&0&2&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&1&0\\0&0&1&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

So $x = 0$, $y = 0$, $z = 0$ is the only solution.

$\vec{v}_1, \ldots, \vec{v}_m$: linearly independent if and only if rank $= m$.

Example: Vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$ are linearly dependent.

$$\vec{0} = x \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} + y \cdot \begin{bmatrix}4\\5\\6\end{bmatrix} + z \cdot \begin{bmatrix}7\\8\\9\end{bmatrix} \quad \ldots \quad \begin{array}{l} x = \cancel{0} -1 \\ y = 2 \\ z = 1 \end{array}$$

Def: The vectors $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$ form a basis of V if $V = \text{span}(\vec{v}_1, \ldots, \vec{v}_m)$ and $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent.

Example:

1) $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$, $\mathbb{R}^3$

0) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$, $\mathbb{R}^3$

2) $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$ Yes! But not $\mathbb{R}^4$! A subspace of $\mathbb{R}^4$!

Non-example: $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$ $\mathbb{R}^3$

To construct a basis of $\text{Im}(A)$ or $\ker(A)$, find [redacted] vectors spanning $\text{Im}(A)$ or $\ker(A)$, and remove the redundant ones.