Change of basis.

Different basis give different expressions of vectors:

$$\mathbb{R}^3: \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \begin{bmatrix}a\\b\\c\end{bmatrix} = a\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} + b\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} + c\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\mathbb{R}^3: \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \qquad \begin{bmatrix}a\\b\\c\end{bmatrix} = (2a-b)\begin{bmatrix}1\\1\\0\end{bmatrix} + (2c-a)\begin{bmatrix}1\\0\\1\end{bmatrix} + (2b-c)\begin{bmatrix}0\\1\\1\end{bmatrix}.$$

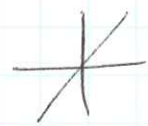
How do we find these scalars? We solve:

$$\left[\begin{array}{ccc|c}1&1&0&a\\1&0&1&b\\0&1&1&c\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c}1&1&0&a\\0&-1&1&b-a\\0&1&1&c\end{array}\right] \xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|c}1&1&0&a\\0&1&1&c\\0&-1&1&b-a\end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c}1&0&-1&a-c\\0&1&1&c\\0&-1&1&b-a\end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c}1&0&-1&a-c\\0&1&1&c\\0&0&2&b-a+c\end{array}\right] \xrightarrow{\frac{R_3}{2}}$$

$$\left[\begin{array}{ccc|c}1&0&-1&a-c\\0&1&1&c\\0&0&1&\frac{b-a+c}{2}\end{array}\right] \xrightarrow{R_1+R_3} \left[\begin{array}{ccc|c}1&0&0&\frac{a+b-c}{2}\\0&1&1&c\\0&0&1&\frac{b-a+c}{2}\end{array}\right] \xrightarrow{R_2-R_3}$$

$$\left[\begin{array}{ccc|c}1&0&0&\frac{a+b-c}{2}\\0&1&0&\frac{a-b+c}{2}\\0&0&1&\frac{-a+b+c}{2}\end{array}\right]$$



$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ bad

better: $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$.

Matrix: $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ in $\mathcal{B}$.

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = \frac{a+b-c}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{a-b+c}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} + \frac{-a+b+c}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\begin{tikzcd} x \arrow[r, "T"] & T(x) \\ \vec{x}_{\mathcal{B}} \arrow[u, "S"] \arrow[r, "T_{\mathcal{B}}"] & T(x)_{\mathcal{B}} \arrow[u, "S"'] \end{tikzcd}$$

$$\left[\begin{array}{ccc|c}1&1&0&1\\1&0&1&1\\0&1&1&1\end{array}\right] \rightsquigarrow \begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}$$

coordinates

$$\begin{bmatrix}1\\1\\1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix} \rightsquigarrow \begin{bmatrix}1\\1\\1\end{bmatrix} = \text{—}.$$

Note: $\begin{bmatrix}1\\1\\1\end{bmatrix} = S \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}_{\mathcal{B}}$ for $S = \begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$

Because:

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix} \\ \\ \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \\ \\ \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} | & | & | \end{bmatrix}\begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix} = S\begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix}_{\mathcal{B}}$$

Taking coordinates is linear!

Consider basis: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$.

1) If $\begin{bmatrix}1\\2\end{bmatrix}$, find $\begin{bmatrix}1\\2\end{bmatrix}_{\mathcal{B}}$.
2) If $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}1\\2\end{bmatrix}$, find $\vec{x}$.

1) Solve: $\begin{bmatrix}1\\2\end{bmatrix} = x \cdot \begin{bmatrix}1\\0\end{bmatrix} + y \cdot \begin{bmatrix}1\\1\end{bmatrix}$.

2) By definition:

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}1\\2\end{bmatrix}$ means: $\vec{x} = 1 \cdot \begin{bmatrix}1\\0\end{bmatrix} + 2 \cdot \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$.

Or: $\vec{x} = S \cdot [\vec{x}]_{\mathcal{B}} = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$.

Matrix of linear transformation: $\mathcal{B} = \{v_1, \ldots, v_n\}$.

$$[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \end{bmatrix} [\vec{x}]_{\mathcal{B}}$$