Orthogonal projections and orthogonal basis:

→ orthogonal vectors.

→ length of a vector. unit vector.

Example: $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$ are orthogonal.

$\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\text{not length 1.}}$

Divide by length to get vectors of unit 1: $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$

→ orthonormal vectors: orthogonal and length 1.
We can make any orthogonal vectors into orthonormal vectors.

Example: $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$ are orthogonal.

(*) $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\frac{1}{\sqrt{2}}$ are orthonormal.

Thm: Orthogonal vectors are linearly independent.

(*) Example: $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$ are a basis: span & lin. indep.

Why? angles!

Thm: Projection onto $V = \text{span}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$

is: $\text{proj}_V(\vec{x}) = \vec{x}$ " [redacted] $(\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \cdots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$

Why? We are doing a bunch of little projections, and then we add up the results.

Example: $\mathbb{R}^3 \to \mathbb{R}^3$, projection onto $V = \text{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$.

Compute: $\text{proj}_V\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) =$

1) Find orthonormal basis: $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$.

2) Project:

$$\text{proj}_V\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} + \cdots =$$

$$=$$

$\to$ orthogonal complement: $V^\perp = \{\vec{x} \mid \vec{v}\cdot\vec{x} = 0 \ \forall \vec{v} \in V\}$.

It is the kernel of the orthogonal projection onto V.

Example: Find $V^\perp$.

$$\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = 0$$

$$\begin{cases} x + z = 0 \\ \quad\;\; y = 0 \end{cases} \rightsquigarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-t\\0\\t\end{bmatrix}.$$

Thm:
1) $V^\perp$ is a subspace.
2) $V \cap V^\perp = \{\vec{0}\}$.
3) $\dim(V) + \dim(V^\perp) = n$.
4) $(V^\perp)^\perp = V$.

Pythagoras, Cauchy, Schwarz, in discussion!

Ex: $\mathbb{R}^4 \to \mathbb{R}^4$ proj onto $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$. Find matrix of transformation.