1. Review basis.
2. Review orthonormal basis.
3. Review projections.
4. Computing perpendicular vectors.
5. Computing orthogonal projections.

## 1. Basis.

Given a vector space $\mathbb{R}^3$, a basis is a collection of vectors that spans and is linearly independent.

spans: every vector in $\mathbb{R}^3$ is a linear combination of the vectors in my basis

linearly independent: there are no redundant vectors in my basis. In other words, no vector in my basis is a linear combination of the others.

Example: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ is a basis of $\mathbb{R}^3$.

Check spanning: $x \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} + y \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} + z \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$

solve for $x, y, z$ in terms of $a, b, c$.

Check linear independence: the only solution to the equation

$x \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} + y \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} + z \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ is $x=0, y=0, z=0$.

must be all 0 for linear independence.

## 2. Orthonormal basis.

It is a basis where all vectors have length 1, and they are all orthogonal/perpendicular to each other.

**Example:** $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$ form an orthonormal basis of $\mathbb{R}^3$.

we know

they are all perpendicular to each other:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}=0$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}=\frac{-1+0+1}{2}=0$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}=0$$

and they all have length 1:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}=\frac{1+0+1}{2}=\frac{2}{2}=1. \longrightarrow \text{length } 1.$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}=1. \longrightarrow \text{length } 1.$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}=\frac{1+0+1}{2}=\frac{2}{2}=1. \longrightarrow \text{length } 1.$$

## 3. Orthogonal projection onto a subspace when we know an orthonormal basis.

In $\mathbb{R}^3$, projection of the vector $\begin{bmatrix}a\\b\\c\end{bmatrix}$ onto the subspace

$V=\text{span}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$ is:

$$\text{proj}_V\begin{bmatrix}a\\b\\c\end{bmatrix}=\underbrace{\left(\begin{bmatrix}a & b & c\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\text{projection onto the line given }\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}}+\underbrace{\left(\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\text{projection onto the line given by }\begin{bmatrix}0\\1\\0\end{bmatrix}}.$$

## 4. Finding orthonormal vectors.

**Example 1:** Find all vectors $\vec{v}$ such that $\vec{v}$ is perpendicular to $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$.

Say $\vec{v} = \begin{bmatrix}x\\y\\z\end{bmatrix}$, we want:

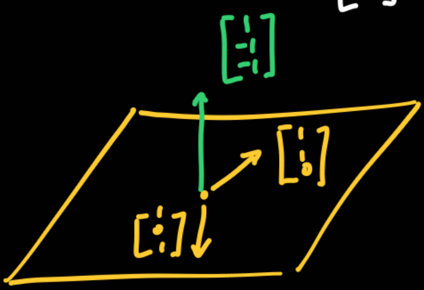
$$\begin{cases} \text{to be perpendicular to the first: } \begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix} = 0 \\ \text{to be perpendicular to the second: } \begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} = 0 \end{cases}$$

$$\begin{cases} x+z=0 \\ x+y=0 \end{cases} \qquad x=t \text{ free, then } \begin{matrix} z=-x \\ y=-x \end{matrix} \qquad \text{so: } \begin{bmatrix}t\\-t\\-t\end{bmatrix} = t\cdot\begin{bmatrix}1\\-1\\-1\end{bmatrix}.$$

Note:

$$t\begin{bmatrix}1\\-1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} = t\cdot(1-1) = 0 \quad \text{and} \quad t\cdot\begin{bmatrix}1\\-1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix} = t\cdot(1-1) = 0.$$

The vectors perpendicular to $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ have the form $t\cdot\begin{bmatrix}1\\-1\\-1\end{bmatrix}$ for $t$ real.



**Example 2:** Find all vectors perpendicular to $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$.

The generic vector in $\mathbb{R}^4$ is: $\begin{bmatrix}w\\x\\y\\z\end{bmatrix}$. Being perpendicular to both means:

$$\begin{cases} \begin{bmatrix}w\\x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = 0 \\ \begin{bmatrix}w\\x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = 0 \end{cases} \qquad \begin{cases} w+y=0 \\ \quad w=-y \\ x+z=0 \\ \quad x=-z \end{cases} \qquad \begin{matrix} y=s \text{ free} \\ z=t \text{ free} \end{matrix} \qquad \begin{bmatrix}-s\\-t\\s\\t\end{bmatrix}.$$

$$\begin{bmatrix}-s\\-t\\s\\t\end{bmatrix} = \begin{bmatrix}-s\\0\\s\\0\end{bmatrix} + \begin{bmatrix}0\\-t\\0\\t\end{bmatrix} = s\cdot\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + t\cdot\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}.$$

The vectors perpendicular to $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$ have the form $s\cdot\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + t\cdot\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$ for $s$ and $t$ real. In particular, all possible linear combinations of $\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$ are the vectors perpendicular to $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$. So:

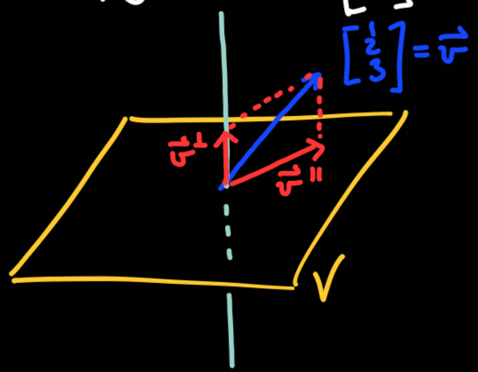
$$\text{span}\left(\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right) \text{ is perpendicular to } \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}.$$

Then:

$$\underbrace{\text{span}\left(\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right)}_{V^\perp} \text{ is perpendicular to } \underbrace{\text{span}\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right)}_{V}.$$

## 5. Orthogonal projections.

<u>Example:</u> Find the projection of $\begin{bmatrix}1\\2\\3\end{bmatrix}$ onto $V = \text{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right)$.




$$\vec{v} = \vec{v}^\parallel + \vec{v}^\perp$$

$$\vec{v}^\parallel = \vec{v} - \vec{v}^\perp$$

Find a vector perpendicular to $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$. We just computed $\begin{bmatrix}1\\-1\\-1\end{bmatrix}$. The unit vector perpendicular to $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$ is $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}$.

We compute:

$$\vec{v}^{\perp} = \mathrm{proj}_{\mathrm{span}\left(\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right)}(\vec{v}) = \left(\begin{bmatrix}1 & 2 & 3\end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix} = \frac{1}{3}\cdot(-4)\cdot\begin{bmatrix}1\\-1\\-1\end{bmatrix} =$$
$$= \frac{4}{3}\cdot\begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

So:

$$\vec{v}^{\parallel} = \vec{v} - \vec{v}^{\perp} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{4}{3}\cdot\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{3}{3}+\frac{4}{3}\\ \frac{6}{3}-\frac{4}{3}\\ \frac{9}{3}-\frac{4}{3}\end{bmatrix} = \begin{bmatrix}\frac{7}{3}\\ \frac{2}{3}\\ \frac{5}{3}\end{bmatrix} = \frac{1}{3}\cdot\begin{bmatrix}7\\2\\5\end{bmatrix}.$$

Allegedly, we computed a vector inside $\mathrm{span}\left(\underline{\begin{bmatrix}1\\0\\1\end{bmatrix}}, \underline{\begin{bmatrix}1\\1\\0\end{bmatrix}}\right)$, this is indeed the case:

$$\frac{1}{3}\begin{bmatrix}7\\2\\5\end{bmatrix} = \frac{5}{3}\cdot\underline{\begin{bmatrix}1\\0\\1\end{bmatrix}} + \frac{2}{3}\cdot\underline{\begin{bmatrix}1\\1\\0\end{bmatrix}}.$$

<u>Example:</u> Find the projection of $\begin{bmatrix}1\\2\\3\end{bmatrix}$ onto $V = \mathrm{span}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\cancel{2}\ \sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}\right)$.

The vectors $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\cancel{2}\ \sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}$ form an orthonormal basis of $V$.

Since $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$ has 0 in the second component, and $\frac{1}{2}\begin{bmatrix}1\\2\\-1\end{bmatrix}$ does <u>not</u> have a 0 in the second component, we cannot write $\frac{1}{\cancel{2}\ \sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}$ as a linear combination of the first. So, the second vector is <u>not</u> redundant.

Thus, these form a basis of $V$.

Also, they are perpendicular: $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\frac{1}{\cancel{2}\ \sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}=\overset{\frac{1}{\sqrt{2\cdot 6}}}{\cancel{\tfrac{1}{2}}}\cdot(1+0-1)=0$

$\frac{1}{2\cdot\sqrt{2}}=\sqrt{2}$

And have length 1:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}=\frac{1}{2}\cdot(1+1)=\frac{2}{2}=1.$$

$$\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}\cdot\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}=\frac{1}{6}\cdot(1+4+1)=\frac{6}{6}=1$$

Now:

$$\mathrm{proj}_V\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)=\left(\begin{bmatrix}1&2&3\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}+\left(\begin{bmatrix}1&2&3\end{bmatrix}\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}=$$

$$=\frac{1}{2}\cdot 4\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}+\frac{1}{6}\cdot 2\cdot\begin{bmatrix}1\\2\\-1\end{bmatrix}=\begin{bmatrix}2\\0\\2\end{bmatrix}+\frac{1}{3}\cdot\begin{bmatrix}1\\2\\-1\end{bmatrix}=$$

$$=\begin{bmatrix}\frac{6}{3}+\frac{1}{3}\\ \frac{2}{3}\\ \frac{6}{3}-\frac{1}{3}\end{bmatrix}=\begin{bmatrix}\frac{7}{3}\\ \frac{2}{3}\\ \frac{5}{3}\end{bmatrix}=\frac{1}{3}\cdot\begin{bmatrix}7\\2\\5\end{bmatrix}.$$

<u>Recommendation:</u> Find the matrix associated to projection onto $V$. $\mathbb{R}^3\longrightarrow\mathbb{R}^3$

Also, find $\underbrace{\text{kernel}}_{\text{will be } V^\perp}$ and $\underbrace{\text{image}}_{\text{will be } V}$.