Orthogonal transformations and orthogonal matrices.

→ linear transformations that preserve length.
orthogonal transformation! $\mathbb{R}^n \to \mathbb{R}^n$.

→ matrix associated to an orthogonal transformation: orthogonal matrix.

Examples: 1) Rotations.
2) Reflections.
} Q: anything similar?

(*)

Thm: A $n \times n$ matrix is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^n$.

(*) These are a special type of change of basis!

Ex: Is: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ orthogonal?

Is: $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix}$ orthogonal?

Is: $\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ orthogonal?

add $\frac{1}{\sqrt{2}}$

Thm: (i) Product of orthogonal matrices is orthogonal.
(ii) Inverse of orthogonal matrices is orthogonal.

Why? They preserve distances! See them as linear transformations.

→ transpose: exchange ij and ji.

Ex: Transpose of $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

→ symmetric matrix: $A^T = A$.

Why? Give 3x3 symmetric matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

symmetric with respect to the diagonal.

→ skew symmetric: $A^T = -A$.

$$\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

Thm: A orthogonal iff $A^T = A^{-1}$.

Q: Is $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} = A$ invertible? Why? Inverse?

$\rightsquigarrow A^{-1} = A^T$.

Helpful for an exercise!
Compute T^{-1} instead of T.

Thm: $V = \mathrm{span}(u_1, \dots, u_m)$. Then matrix of orthogonal projection is =

$$P = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} - & \vec{u}_1 & - \\ & \vdots & \\ - & \vec{u}_m & - \end{bmatrix}.$$

Ex: Projection onto line $y = x$.

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$