Introduction to determinants.

Def: A **determinant** is a function whose input are square matrices and whose output are real numbers:

$$\det : \underbrace{M_n(\mathbb{R})}_{\substack{n\times n \text{ matrices} \\ \text{with real entries}}} \longrightarrow \mathbb{R}$$

$\uparrow$ real numbers

satisfying:

(i) It is linear with respect to each column:

$$\det\begin{bmatrix} | & & | & & | \\ c_1 & \cdots & c_i + c_i' & \cdots & c_n \\ | & & | & & | \end{bmatrix} = \det\begin{bmatrix} | & & | & & | \\ c_1 & \cdots & c_i & \cdots & c_n \\ | & & | & & | \end{bmatrix} + \det\begin{bmatrix} | & & | & & | \\ c_1 & \cdots & c_i' & \cdots & c_n \\ | & & | & & | \end{bmatrix}$$

$$\det\begin{bmatrix} | & & | & & | \\ c_1 & \cdots & k\,c_i & \cdots & c_n \\ | & & | & & | \end{bmatrix} = k\cdot\det\begin{bmatrix} | & & | & & | \\ c_1 & \cdots & c_i & \cdots & c_n \\ | & & | & & | \end{bmatrix}$$

Example:

1) $\det\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \det\begin{bmatrix} 1+2 & 1 \\ 1+1 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \det\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$

2) $\det\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \det\begin{bmatrix} 2\cdot 1 & 1 \\ 2\cdot 1 & 1 \end{bmatrix} = 2\cdot\det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$

(ii) It is alternating in the columns.

$$\det\begin{bmatrix} | & & | & & | & & | \\ c_1 & \cdots & c_i & \cdots & c_i & \cdots & c_n \\ | & & | & & | & & | \end{bmatrix} = 0.$$

Example: $\det\begin{bmatrix}1&1\\1&1\end{bmatrix}=0.$ $\qquad\det\begin{bmatrix}1&2&2\\1&3&3\\1&4&4\end{bmatrix}=0$

(iii) The determinant of the identity is 1.

$$\det\begin{bmatrix}1&&0\\&\ddots&\\0&&1\end{bmatrix}=1.$$

Theorem: A determinant exists, and it is unique.

Example: What is the determinant of the 2×2 matrix $\begin{bmatrix}a&b\\c&d\end{bmatrix}$?

It is:

$$\det\begin{bmatrix}a&b\\c&d\end{bmatrix}=ad-bc.$$

Exercise: check $\det\begin{bmatrix}a+a'&b\\c+c'&d\end{bmatrix}=\det\begin{bmatrix}a&b\\c&d\end{bmatrix}+\det\begin{bmatrix}a'&b\\c'&d\end{bmatrix}.$

In general, to compute determinants, we use the cofactor expansion.

Example: Find the determinant of $\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix}$.

Step 1: choose a column. Say column 2.

$$(-1)^{1+2}\cdot 2\cdot\det\begin{bmatrix}3&1\\2&2\end{bmatrix}+(-1)^{2+2}\cdot 2\cdot\det\begin{bmatrix}1&3\\2&2\end{bmatrix}+(-1)^{3+2}\cdot 1\cdot\det\begin{bmatrix}1&3\\3&1\end{bmatrix}=$$

$$\overset{1,2}{\begin{bmatrix}\cancel{1}&2&\cancel{3}\\3&\cancel{2}&1\\2&\cancel{1}&2\end{bmatrix}}\qquad\overset{2,2}{\begin{bmatrix}1&\cancel{2}&3\\\cancel{3}&2&\cancel{1}\\2&\cancel{1}&2\end{bmatrix}}\qquad\overset{3,2}{\begin{bmatrix}1&\cancel{2}&3\\3&\cancel{2}&1\\\cancel{2}&1&\cancel{2}\end{bmatrix}}$$

$$= -2\cdot(3\cdot 2-1\cdot 2)+2\cdot(1\cdot 2-3\cdot 2)-1\cdot(1\cdot 1-3\cdot 3)=$$

$$= -2\cdot 4+2\cdot(-4)-(-8)=-8.$$

So: $\det\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix}=-8.$ We computed the expansion along column 2.

Compute expanding along a row:

choose 3rd row.

$$\det\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix}=(-1)^{3+1}\cdot 2\cdot\det\begin{bmatrix}2&3\\2&1\end{bmatrix}+(-1)^{3+2}\cdot 1\cdot\det\begin{bmatrix}1&3\\3&1\end{bmatrix}+(-1)^{3+3}\cdot 2\cdot\det\begin{bmatrix}1&2\\3&2\end{bmatrix}=$$

$$\overset{3,1}{\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix}}\qquad\overset{3,2}{\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix}}\qquad\overset{3,3}{\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix}}$$

$$=2\cdot(2\cdot 1-3\cdot 2)-1\cdot(1\cdot 1-3\cdot 3)+2\cdot(1\cdot 2-2\cdot 3)=$$

$$=2\cdot(-4)-(-8)+2\cdot(-4)=-8.$$

Theorem: The determinant of a triangular matrix is the multiplication of its diagonal entries.

Example: Compute:

expand 1st column

$$\det\begin{bmatrix}1&*&*&*&*\\0&2&*&*&*\\0&0&3&*&*\\0&0&0&4&*\\0&0&0&0&5\end{bmatrix}=1\cdot\det\begin{bmatrix}2&*&*&*\\0&3&*&*\\0&0&4&*\\0&0&0&5\end{bmatrix}=1\cdot 2\cdot\det\begin{bmatrix}3&*&*\\0&4&*\\0&0&5\end{bmatrix}=$$

$$= 1\cdot 2\cdot 3\cdot \det\underbrace{\begin{bmatrix} 4 & * \\ 0 & 5 \end{bmatrix}}_{} = 1\cdot 2\cdot 3\cdot \underline{4\cdot 5}.$$

<u>Theorem:</u> The determinant of the transpose is equal to the determinant of the original matrix: $\det(A^T) = \det(A)$.

<u>Why?</u> Because expanding can be done by columns or rows!

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \text{expand along the first column} = -8.$$

$$\det\begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \text{expand along the first row} = -8.$$

$$3\cdot \det\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$\updownarrow$ transposes

$$3\cdot \det\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

transposes don't change the determinant of 2×2 matrices!

<u>Theorem:</u> A matrix is invertible if and only if its determinant is <u>not</u> zero.