Finding the eigenvalues for a matrix:

Recall: → eigenvalues
→ eigenvectors
→ eigenbasis.

Thm: A $n \times n$ diagonalizable iff there is eigenbasis of $\mathbb{R}^n$ for A.
$S^{-1}AS = D$

To compute eigenbasis we first compute eigenvectors and then we compute eigenvalues.

Thm: 1) λ is eigenvalue of A iff $\det(A - \lambda \cdot I_n) = 0$.
2) The eigenvalues of A are the solutions λ to the equation $\det(A - \lambda \cdot I_n) = 0$.

Def: This is called the characteristic equation.
The polynomial $\det(A - \lambda \cdot I_n)$ is called the characteristic poly.

Ex: Find eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A$.

$$\det(A - \lambda \cdot I_n) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} =$$
$$= (1-\lambda)^2 - 1 = 1 + \lambda^2 - 2\lambda - 1 = -2\lambda + \lambda^2 = \lambda(\lambda - 2).$$

Roots: $\lambda = 0$ and $\lambda = 2$.

Ex: Find eigenvalues of $\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

$$p_A(x) = \det \frac{1}{3}\begin{bmatrix} 2-3\lambda & -1 & -1 \\ -1 & 2-3\lambda & -1 \\ -1 & -1 & 2-3\lambda \end{bmatrix} = \frac{1}{27}\left((2-3\lambda) \cdot \det \begin{bmatrix} 2-3\lambda & -1 \\ -1 & 2-3\lambda \end{bmatrix}\right.$$
$$\left. + \det \begin{bmatrix} -1 & -1 \\ -1 & 2-3\lambda \end{bmatrix} - \det \begin{bmatrix} -1 & -1 \\ 2-3\lambda & -1 \end{bmatrix}\right) = \frac{1}{27} \cdot \Big((2-3\lambda) \cdot$$
$$((2-3\lambda)^2 - 1) + (3\lambda - 2) - 1 - 1 - 2 + 3\lambda\Big) = \frac{1}{27}\Big((2-3\lambda) \cdot$$
$$(4 + 9\lambda^2 - 12\lambda - 1) + 6\lambda - 6\Big) = \frac{1}{27} \cdot \big(6 + 18\lambda^2 - 24\lambda - 9\lambda - 27\lambda^3 + 36\lambda^2 + 6\lambda - 6\big) = (*)$$

$$(*) = \frac{1}{27}\left(-27\lambda^3 + 54\lambda^2 - 27\lambda\right) =$$
$$-\lambda^3 + 2\lambda^2 - \lambda =$$
$$= \lambda(-\lambda^2 + 2\lambda - 1)$$
$\lambda = 0$ $\lambda = 1$

Rmk: The characteristic poly of $n \times n$ matrix has degree n.

Def: Algebraic multiplicity = maximum number of times we can factor a root: $p_A(x) = (x-\lambda)^k \cdot g(x)$, $g(\lambda) \neq 0$.

$$p_A(x) = f_A(x)$$

So $p_A(x)$ has at most n distinct eigenvalues.

If n odd then $p_A(x)$ has at least one eigenvalue.

Q: Matrix with no real eigenvalues: rotations.

Def: Trace.

Thm: Let $p_A(x) = (x-\lambda_1)\cdots(x-\lambda_n) \cdot c$,

then:

$$\det(A) = \lambda_1 \cdots \lambda_n \quad \text{and} \quad tr(A) = \lambda_1 + \cdots + \lambda_n.$$

Why?

$$\det(S^{-1}AS) = \det(D)$$

top coefficient in $p_A(x) = f_A(x)$.