Quadratic forms: (24)

Def: A quadratic form is a polynomial of degree 2 in n variables, where each term has exactly degree 2. $q(x_1, \dots, x_n)$.

Ex: $q(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$

Note: $q(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = x_1^2 + x_1x_2 + x_1x_2 + x_2^2.$

We can always do this! For: (projection $x+y+z=0$) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

we have $q(x_1, x_2, x_3) = \frac{2}{3} x_1^2 + \frac{2}{3} x_2^2 + \frac{2}{3} x_3^2 - \frac{2}{3} x_1x_2 - \frac{2}{3} x_2x_3 - \frac{2}{3} x_1x_3$

$q(x_1, \dots, x_n) = \vec{x}^T A \vec{x}$. for A symmetric.

We can now use the Spectral Theorem to simplify $q(x_1, \dots, x_n)$.

Recall: A orthogonally diagonalizable if and only if A symmetric.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. $\quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\mathcal{B} = \Big\{ \underbrace{\tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{u}_1}, \underbrace{\tfrac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{u}_2} \Big\}.$$

$$q(x_1, x_2) = (c_1 \vec{u}_1^T + c_2 \vec{u}_2^T) A (c_1 \vec{u}_1 + c_2 \vec{u}_2) =$$
$$= (c_1 \vec{u}_1^T + c_2 \vec{u}_2^T)(c_1 2 \vec{u}_1 + c_2 \cdot 0 \vec{u}_2) =$$
$$= 2 c_1^2 + 0 \cdot c_2^2 \quad \text{where } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Thm: A symmetric, $q(x_1, x_2) = \vec{x}^T A \vec{x}$, $\mathcal{B}$ orthonormal basis of $\mathbb{R}^2$ for A with eigenvalues λ_1, λ_2 (with multiplicity), then: $q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2$ with $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

The signs of λ_1 and λ_2 are extremely important!
They can help in solving max-min problems at $q(0,0)$!

Def: A symmetric, $q(\vec{x}) = \vec{x}^T A \vec{x}$. A positive definite if $\underbrace{q(\vec{x})}_{>0}$ positive for all $\vec{x} \neq 0$. A positive semidefinite $q(\vec{x}) \geq 0$ for all $\vec{x}$. A indefinite if $q(\vec{x})$ is positive and negative.

Thm: A positive definite if and only if positive eigenvalues. A positive semidefinite if and only if positive or zero eigenvalues.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is positive semidefinite.

Applications: physics: $\langle \psi | A | \psi \rangle$ A operator.
probability: $\|A \cdot \vec{x}\|^2$ is quadratic form.

Singular values: Finding orthonormal vectors that remain orthogonal after doing a transformation.
A $n \times m$

Def: Singular values are the square roots of the eigenvalues of $A^T A$; with algebraic multiplicities.

$$\sigma = \sqrt{\lambda}$$

singular value A — eigenvalue $A^T A$.

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

A $n \times n$:

λ_1 $\vec{v}_1$
$\vee\!\!/$
$\vdots$
$\vee\!\!/$
λ_n v_n

descending order

eigenbasis $A^T A$

$\sigma_1^2 = \lambda_1$ $\quad$ $u_1 = \frac{1}{\sigma_1} A \vec{v}_1$
$\vdots$
$\sigma_n^2 = \lambda_n$ $\quad$ $u_n = \frac{1}{\sigma_n} A \cdot v_n$

$$A = U \Sigma V^T$$

$U = \begin{bmatrix} | \\ u_i \\ | \end{bmatrix}$ $\quad$ $V^T = \begin{bmatrix} - v_i - \end{bmatrix}$

Examples in the class notes.