Math 115A
Linear Algebra

Discussion 3

Problem 1.

Let V and W be vector spaces, let $T : V \to W$ be a linear function.

(a) Prove that T is injective if and only if T sends linearly independent subsets of V to linearly independent subsets of W.

(b) Suppose that T is injective and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c) Suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for V and T is injective and surjective. Prove that $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a basis of W.

Problem 2($\star$).

(a) Prove that the function

$$\begin{array}{cccc} T: & \mathbb{F}[x] & \longrightarrow & \mathbb{F}[x] \\ & f(x) & \longmapsto & \int_0^x f(t)dt \end{array}$$

is linear and injective, but not surjective.

(b) Prove that the function

$$\begin{array}{cccc} T: & \mathbb{F}[x] & \longrightarrow & \mathbb{F}[x] \\ & f(x) & \longmapsto & \frac{d}{dx} f(x) \end{array}$$

is linear and surjective, but not injective.

Problem 3.

Let $\mathbb{F}$ be a field and $V = \{\{a_i\}_{i\in\mathbb{N}} | a_i \in \mathbb{F} \text{ and } \exists n \in \mathbb{N} \text{ with } a_i = 0 \text{ for all } i \geq n\}$.

(a) Prove that V is a vector space with $\{a_i\}_{i\in\mathbb{N}} + \{b_i\}_{i\in\mathbb{N}} = \{a_i + b_i\}_{i\in\mathbb{N}}$ and $c\{a_i\}_{i\in\mathbb{N}} = \{ca_i\}_{i\in\mathbb{N}}$ for all $\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}} \in V$ and $c \in \mathbb{F}$.

(b) Prove that the function $L : V \to V$ given by $L(\{a_i\}_{i\in\mathbb{N}}) = \{a_{i+1}\}_{i\in\mathbb{N}}$ is linear and surjective, but not injective.

(c) Prove that the function $R : V \to V$ given by $R(\{a_i\}_{i\in\mathbb{N}}) = \{a_{i-1}\}_{i\in\mathbb{N}}$, where $a_{-1} = 0$ by convention, is linear and injective, but not surjective.

Problem 4($\star$).

Let V and W be finite-dimensional vector spaces and $T : V \to W$ be a linear function.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be surjective.

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be injective.

Problem 5.

Let V be a vector space, let $T : V \to V$ be a linear function. Given a subset U of V we say that U is *T-invariant* if $T(U) \subseteq U$. Prove that if $U_1, U_2 \subseteq V$ are T-invariant, then $U_1 + U_2$ is T-invariant.

Problem 6.

Let $T : V \to V$ be a linear function. Prove that $\{0\}$, V, $\ker(T)$, and $\operatorname{im}(T)$ are all T-invariant.

Problem 7.

Let $T : V \to V$ be a linear function and U a T-invariant subset of V. Define the restriction of T on U as the function $T_U : U \to U$ given by $T_U(u) = T(u)$ for all $u \in U$. Prove that $T_U : U \to U$ is linear.