# Math 115A
# Linear Algebra

## Discussion 4

**Problem 1.**

Let $V$ be a finite dimensional vector space with an ordered basis $\beta$. Define the function $T : V \to \mathbb{F}^n$ by $T(x) = [x]_\beta$. Prove that $T$ is linear.

**Problem 2.**

A function $T : V \to W$ between the vector spaces $V$ and $W$ is called *additive* when $T(x+y) = T(x) + T(y)$ for all $x, y \in V$. Let $V = \mathbb{C}$ be the vector space of complex numbers over the field $\mathbb{C}$. Define the function $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of $z$.

(a) Prove that $T$ is additive.

(b) Prove that $T$ is not linear.

**Problem 3($\star$).**

Let $V = \mathbb{C}$ be the vector space of complex numbers over the field $\mathbb{R}$. Define the function $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of $z$.

(a) Prove that $T$ is additive.

(b) Prove that $T$ is linear.

(c) Let $\beta = \{1, i\}$. Prove that $\beta$ is a basis of $V$ over $\mathbb{R}$.

(d) Compute $[T]_\beta$.

**Problem 4.**

Let $V$ be a vector space with the ordered basis $\beta = \{v_1, \ldots, v_n\}$. Let $v_0 = 0$, let $T : V \to V$ be a linear transformation such that $T(v_j) = v_j + v_{j-1}$ for $j \in \{1, \ldots, n\}$.

(a) Prove that $T$ exists and that $T$ is unique.

(b) Compute $[T]_\beta$.

**Problem 5($\star$).**

Let $V$ be a vector space of dimension $n$, let $T : V \to V$ be a linear function. Suppose that $W$ is a $T$-invariant subspace of $V$ with dimension $k$. Show that there exists a basis $\beta$ of $V$ such that

$$[T]_\beta = \begin{bmatrix} A & B \\ O & C \end{bmatrix}$$

where $A$ is a $k \times k$ matrix, $B$ and $C$ are $k \times (n-k)$ matrices, and $O$ is the $(n-k) \times k$ zero matrix.

**Problem 6.**

Let $A$ be an $n \times n$ matrix. Prove that $A$ is a diagonal matrix if and only if $A_{ij} = \delta_{ij} A_{ij}$ for all $i, j \in \{1, \ldots, n\}$.

**Problem 7.**

Let $A$ and $B$ be $n \times n$ matrices. The *trace* of a matrix $A$, denoted $\operatorname{tr}(A)$, is the sum of its diagonal entries. Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Prove that $\operatorname{tr}(A) = \operatorname{tr}(A^t)$.