Math 115A
Linear Algebra

Practice Final

Instructions: You will have 3 hours to complete the exam. There will be 8 questions, worth a total of 100 points. There will be 1 true or false question, and there will be 7 short answer questions. This test will be closed book and closed notes. No calculator will be allowed. Please write your solutions in the space provided, show all your work legibly, and clearly reference any theorems or results that you use. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Once the 3 hours have elapsed, you are not allowed to continue writing and you are not allowed to communicate with anybody except the administrators of the exam. Please follow their requests at all times. Failure to comply with any of these instructions may have repercussions in your final grade. Here you have a list of problems to practice for the exam.

Name: ______________________________

ID number: ______________________________

Section: ______________________________

Problem 1: True or False.

Determine whether the following statements are true or false. If the statement is true, write **T** over the line provided before the statement. If the statement is false, write **F** over the line provided before the statement. Do NOT write "true" or "false".

(a) ___ For all values $p \in \mathbb{N}$ we have that $\mathbb{Z}_p$ is a field.

(b) ___ Let V be a vector space and $U_1, U_2, U_3 \subseteq V$ be vector subspaces. If $V = U_1 \oplus U_2$ and $V = U_1 \oplus U_3$ then $U_2 = U_3$.

(c) ___ Let V be a vector space, let $v \in V$ be such that $v + w = w$ for all $w \in V$. Then it is possible that $w \neq 0_V$.

(d) ___ Let S be a subset of V a vector space. Then S is linearly independent if and only if each finite subset of S is linearly independent.

(e) ___ Let V be a vector space of dimension n, let S be a subset of V with n elements. Then S is linearly independent if and only if the span of S is V.

(f) ___ There exists a vector space V, a basis β of V, and an element $v \in V$ such that the decomposition of v as a linear combination of the elements in β is not unique.

(g) ___ Let $U \in \mathcal{L}(V, W)$ with β a finite basis of V and γ a finite basis of W. Then $U = T_{[U]_\beta^\gamma}$.

(h) ___ Let $A \in M_{n\times n}(\mathbb{F})$. Then A is invertible if and only if T_A is invertible.

(i) ___ Let $A \in M_{n\times n}(\mathbb{F})$. Then T_A is invertible.

(j) ___ Let $A, B \in M_{n\times n}(\mathbb{F})$. If A is similar to B then there exists a matrix $Q \in M_{n\times n}(\mathbb{F})$ such that $B = Q^t AQ$.

(k) ___ Let $A, B \in M_{n\times n}(\mathbb{F})$. If A is similar to B then B is similar to A.

(l) ___ Let $T \in \mathcal{L}(V, W)$. Then $\ker(T)$ is a vector subspace of V.

(m) ___ Let $T \in \mathcal{L}(V, W)$. Then $\mathrm{Im}(T)$ is a vector subspace of W.

(n) ___ Let V be a vector space of dimension n and $T \in \mathcal{L}(V, V)$. Then there exists a polynomial $g(t)$ of degree n such that $g(T) = 0$.

(o) ___ Let V be a finite dimensional vector space, let $T \in \mathcal{L}(V, V)$, and let $g \in \mathbb{F}[x]$. Then $g(T) \neq 0$.

(p) ___ Let $T \in \mathcal{L}(V, W)$ have eigenvalues λ_1 and λ_2. Then T has eigenvalue $\lambda_1 + \lambda_2$.

(q) ___ Let $T \in \mathcal{L}(V, W)$ have eigenvalues λ and μ. Then T has eigenvalue $\lambda \cdot \mu$.

(r) ___ Let V be a vector space of dimension n and $T \in \mathcal{L}(V, V)$. If T has less than n distinct eigenvalues then T is not diagonalizable.

(s) ___ Let V be a vector space of dimension n and $T \in \mathcal{L}(V, V)$. If T does not have eigenvalues then T is not diagonalizable.

(t) ___ Let V be a vector space and $S \subseteq V$ be an orthogonal subset of V. Then S is linearly independent.

(u) ___ Let V be an inner product vector space, then $||v|| \geq 0$ for all $v \in V$.

Problem 2. *1pts.*

Let V be a finite dimensional vector space over a field $\mathbb{F}$, let $T \in \mathcal{L}(V, V)$.

(a) Suppose that $\dim(\mathrm{im}(T)) = \dim(\mathrm{im}(T^2))$. Prove that $V = \ker(T) \oplus \mathrm{im}(T)$.

(b) Prove that there exists a non-zero $k \in \mathbb{N}$ such that $\dim(\mathrm{im}(T^k)) = \dim(\mathrm{im}(T^{k+1}))$.

(c) Prove that there exists a non-zero $k \in \mathbb{N}$ such that $V = \ker(T^k) \oplus \mathrm{im}(T^k)$.

Problem 3. *1pts.*

Let V be a finite dimensional vector space over a field $\mathbb{F}$, let $T \in \mathcal{L}(V, \mathbb{F})$, let $v \in V$ and $v \notin \ker(T)$. Prove that $V = \ker(T) \oplus \mathrm{span}(v)$, where $\mathrm{span}(v) = \{cv | c \in \mathbb{F}\}$.

Problem 4. *1pts.*

Let V be a finite dimensional vector space over a field $\mathbb{F}$, let $T \in \mathcal{L}(V, V)$.

(a) Suppose that $V = \mathrm{im}(T) + \ker(T)$. Prove that $V = \mathrm{im}(T) \oplus \ker(T)$.

(b) Suppose that $\mathrm{im}(T) \cap \ker(T) = \{0\}$. Prove that $V = \mathrm{im}(T) \oplus \ker(T)$.

Problem 5. *1pts.*

Let V and W be vector spaces over a field $\mathbb{F}$, let S be a subset of V, define $S^0 = \{T \in \mathcal{L}(V, W) | T(x) = 0 \text{ for all } x \in S\}$.

(a) Show that S^0 is a subspace of $\mathcal{L}(V, W)$.

(b) If S_1 and S_2 are subsets of V with $S_1 \subseteq S_2$ show that $S_2^0 \subseteq S_1^0$.

(c) If V_1 and V_2 are vector subspaces of V show that $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

Problem 6. *1pts.*

Let V and W be finite dimensional vector spaces over a field $\mathbb{F}$, let $T \in \mathcal{L}(V, W)$. Prove that the following are equivalent.

(a) T is injective.

(b) If S is a linearly independent subset of V then $T(S)$ is a linearly independent subset.

Problem 7. *1pts.*

Let V and W be finite dimensional vector spaces over a field $\mathbb{F}$, let $T \in \mathcal{L}(V, W)$. Prove that the following are equivalent.

(a) T is injective.

(b) $\ker(T) = \{0\}$.

Problem 8. *1pts.*

Let V and W be finite dimensional vector spaces of the same dimension over a field $\mathbb{F}$, let $T \in \mathcal{L}(V, W)$. Prove that the following are equivalent.

(a) T is injective.

(b) T is surjective.

Problem 9. *1pts.*

Let $A \in M_{n\times m}$, $B \in M_{m\times p}(\mathbb{F})$, $C \in M_{p\times q}(\mathbb{F})$. Show that $(AB)C = A(BC)$ using only the definition of matrix multiplication.

Problem 10. *1pts.*

(a) Let V be a finite dimensional vector space over a field $\mathbb{F}$, let $A, B \in \mathcal{L}(V, V)$ such that TS is invertible. Show that T and S are invertible.

(b) Let $A, B \in M_{n\times n}(\mathbb{F})$ such that AB is invertible. Show that A and B are invertible.

Problem 11. *1pts.*

Let V and W be finite dimensional vector spaces over $\mathbb{F}$, let $T \in \mathcal{L}(V, W)$. Show that T is an isomorphism if and only if T is injective and surjective.

Problem 12. *1pts.*

State and prove the Replacement Theorem.

Problem 13. *1pts.*

Let V be a finite dimensional vector space over a field $\mathbb{F}$ with basis $\beta = \{v_1, \dots, v_n\}$, let $Q \in M_{n\times n}(\mathbb{F})$ be invertible. For all $j \in \{1, \dots, n\}$ let

$$w_j = \sum_{i=1}^{n} Q_{ij} v_i = Q_{1j} v_1 + \cdots + Q_{nj} v_n.$$

and $\gamma = \{w_1, \dots, w_n\}$.

(a) Prove that $\gamma = \{w_1, \dots, w_n\}$ is a basis of V.

(b) Prove that $Q = [\mathrm{id}_V]_\gamma^\beta$.

Problem 14. *1pts.*

Let V be a finite dimensional vector space over a field $\mathbb{F}$ with basis $\beta = \{v_1, \dots, v_n\}$, let $T \in \mathcal{L}(V, V)$. Prove that the following are equivalent:

(a) The matrix $[T]_\beta^\beta$ is upper triangular.

(b) $T(v_i) \in \mathrm{span}(v_1, \dots, v_i)$ for all $i = 1, \dots, n$.

(c) $\mathrm{span}(v_1, \dots, v_i)$ is T-invariant for all $i = 1, \dots, n$.

Problem 15. *1pts.*

Let V be a finite dimensional vector space, let $\dim_{\mathbb{F}}(V) = n$, let β be an ordered basis of V. Define the function $\phi : V \to \mathbb{F}^n$ by $\phi(x) = [x]_\beta$.

(a) Prove that ϕ is linear.

(b) Prove that ϕ is injective.

(c) Prove that ϕ is surjective.

(d) Prove that ϕ is an isomorphism.

Problem 16. *1pts.*

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let β and γ be ordered bases for V and W, respectively. Let $\dim_{\mathbb{F}}(V) = n$ and $\dim_{\mathbb{F}}(W) = m$. Define the function $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(\mathbb{F})$ by $\Phi(T) = [T]_\beta^\gamma$ for each $T \in \mathcal{L}(V, W)$.

(a) Prove that Φ is linear.

(b) Prove that Φ is injective.

(c) Prove that Φ is surjective.

(d) Prove that Φ is an isomorphism.

Problem 17. *1pts.*

Let V and W be finite dimensional vector spaces over a field $\mathbb{F}$, let $T \in \mathcal{L}(V, W)$. Prove that if $\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(W)$ then there exists a basis β of V and a basis γ of W such that $[T]_\beta^\gamma$ is a diagonal matrix.

Problem 18. *1pts.*

Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *even* if $f(t) = f(-t)$ for all $t \in \mathbb{R}$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *odd* if $f(t) = -f(-t)$ for all $t \in \mathbb{R}$.

(a) Prove that the space of even functions $\mathcal{E}$ is a vector subspace of V.

(b) Prove that the space of odd functions $\mathcal{O}$ is a vector subspace of V.

(c) Prove that $V = \mathcal{E} \oplus \mathcal{O}$.

Problem 19. *1pts.*

Let V be a finite dimensional vector space over $\mathbb{F}$ of dimension n. Suppose that $T \in \mathcal{L}(V, V)$ has n distinct eigenvalues. Show that T has n distinct eigenvectors forming a basis of V.

Problem 20. *1pts.*

Let V be a finite dimensional vector space over $\mathbb{R}$. Show that if $\dim_{\mathbb{R}}(V)$ is odd, then every $T \in \mathcal{L}(V, V)$ has an eigenvalue.

Problem 21. *1pts.*

Let V be a finite dimensional vector space over a field $\mathbb{F}$, let $T, S \in \mathcal{L}(V, V)$ be diagonalizable.

(a) Define *diagonalizable* operator.

(b) Define *simultaneously diagonalizable* operators.

(c) Prove that T and S are simultaneously diagonalizable if and only if $ST = TS$.

Problem 22. *1pts.*

Let V be a finite dimensional vector space over a field $\mathbb{F}$, let $T \in \mathcal{L}(V, V)$.

(a) Prove that $\{\mathrm{id}_V, T, T^2, \ldots, T^{n^2}\}$ is a linearly dependent subset of $\mathcal{L}(V, V)$.

(b) Prove that there exists a polynomial $f \in \mathbb{F}[x]$ of degree n^2 such that $f(T) = 0$.

Problem 23. *1pts.*

Let V be a finite dimensional vector space over $\mathbb{F}$, let $T \in \mathcal{L}(V, V)$ be invertible.

(a) Prove that if v is an eigenvector of T with corresponding eigenvalue λ, then v is an eigenvector of T^{-1} with corresponding eigenvalue λ^{-1}.

(b) Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(c) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Problem 24. *1pts.*

Let V and W be vector spaces over $\mathbb{F}$, let $T \in \mathcal{L}(V, W)$.

(a) Prove that $\ker(T)$ is a vector subspace of V.

(b) Prove that $\operatorname{im}(T)$ is a vector subspace of W.

(c) Prove that $\phi : V/\ker(T) \to \operatorname{im}(T)$ given by $\phi(v + \ker(T)) = T(v)$ for all $v + \ker(T) \in V/\ker(T)$ is well defined. That is, prove that if $v_1, v_2 \in V$ are such that $v_1 + \ker(T) = v_2 + \ker(T)$, then $\phi(v_1 + \ker(T)) = \phi(v_2 + \ker(T))$.

(d) Prove that ϕ is injective.

(e) Prove that ϕ is surjective.

(f) Prove that $V/\ker(T)$ is isomorphic to $\operatorname{im}(T)$.

Note that V and W are not assumed to be finite dimensional. If you would like to prove the statement adding the hypothesis that V and W are finite dimensional, you will be awarded partial credit.

Problem 25. *1pts.*

Let V be a finite dimensional vector spaces over $\mathbb{F}$, let β be a basis of V, let $\beta_1, \ldots, \beta_k$ be subsets of β such that $\beta = \beta_1 \cup \cdots \cup \beta_k$ and $\beta_i \cap \beta_j = \emptyset$ for all distinct $i, j = 1, \ldots, k$. Show that $V = \operatorname{span}(\beta_1) \oplus \cdots \oplus \operatorname{span}(\beta_k)$.

Problem 26. *1pts.*

Let $A \in M_{n\times n}(F)$.

(a) Prove that A and A^t have the same characteristic polynomial.

(b) Prove that A and A^t have the same eigenvalues.

(c) Let $\lambda \in \mathbb{F}$ be an eigenvalue of A. Let E_λ be the eigenspace of A corresponding to λ, let E^t_λ be the eigenspace of A^t corresponding to λ. Prove that $\dim_{\mathbb{F}}(E_\lambda) = \dim_{\mathbb{F}}(E^t_\lambda)$.

(d) Prove that A is diagonalizable if and only if A^t is diagonalizable.