<u>Theorem:</u> (Rank-Nullity) Let $T: V \to W$ be a linear transformation, let V be finite dimensional. Then:

$$\dim(V) = \underbrace{\dim(\ker(T))}_{\text{nullity of } T} + \underbrace{\dim(\mathrm{Im}(T))}_{\text{rank of } T}.$$

<u>Proof:</u> Since V is finite dimensional, say $\dim(V) = n$ and $\{v_1, \dots, v_n\}$ is a basis of V.

Since $\ker(T)$ is a vector subspace of V, then $\dim(\ker(T)) = k \le n$.

Consider $\{w_1, \dots, w_k\}$ a basis of $\ker(T)$. Extend it to a basis of V:

$$\underbrace{\{\overbrace{w_1, \dots, w_k}^{k = \dim(\ker(T))}, \underbrace{w_{k+1}, \dots, w_n}_{n-k \;\rightsquigarrow\; \text{this should be } \dim(\mathrm{Im}(T))}\}}_{n = \dim(V)}.$$

Maybe applying T to $w_{k+1}, \dots, w_n$ gives a basis of $\mathrm{Im}(T)$.

We want to prove that $\{\underbrace{T(w_{k+1}), \dots, T(w_n)}_{n-k}\}$ form a basis of $\mathrm{Im}(T)$.

We first prove that $\mathrm{Im}(T) = \mathrm{Span}\{T(w_{k+1}), \dots, T(w_n)\}$.

$\supseteq$) Since $T(w_{k+1}), \dots, T(w_n) \in \mathrm{Im}(T)$ then

$$\mathrm{Span}\{T(w_{k+1}), \dots, T(w_n)\} \subseteq \mathrm{Im}(T).$$

$\subseteq$) An element in $\mathrm{Im}(T)$ has the form $T(v)$ for some $v \in V$.

Since $\{w_1, \dots, w_n\}$ form a basis of V then:

$v = a_1 w_1 + \dots + a_n w_n$ for some $a_1, \dots, a_n \in \mathbb{F}$.

$T(v) = T(a_1 w_1 + \dots + a_n w_n) =$

$$= a_1 \underbrace{T(w_1)}_{=0} + \cdots + a_k \cdot \underbrace{T(w_k)}_{=0} + a_{k+1} \cdot T(w_{k+1}) + \cdots + a_n \cdot T(w_n) =$$

$$= a_{k+1} \cdot T(w_{k+1}) + \cdots + a_n \cdot T(w_n) \in \operatorname{Span}\{T(w_{k+1}), \dots, T(w_n)\}.$$

Since $T(v) \in \operatorname{Span}\{T(w_{k+1}), \dots, T(w_n)\}$ then

$$\operatorname{Im}(T) \subseteq \operatorname{Span}\{T(w_{k+1}), \dots, T(w_n)\}.$$

We now prove that $\{T(w_{k+1}), \dots, T(w_n)\}$ is linearly independent. Assume it is not, namely there are some non-zero scalars $a_{k+1}, \dots, a_n \in \mathbb{F}$ such that:

$$a_{k+1} \cdot T(w_{k+1}) + \cdots + a_n \cdot T(w_n) = 0$$

$$T(\underbrace{a_{k+1} \cdot w_{k+1} + \cdots + a_n \cdot w_n}_{\text{vector in } V}) = 0$$

Thus $a_{k+1} \cdot w_{k+1} + \cdots + a_n \cdot w_n \in \operatorname{Ker}(T)$. Then:

$$a_{k+1} \cdot w_{k+1} + \cdots + a_n \cdot w_n = a_1 \cdot w_1 + \cdots + a_k \cdot w_k$$

$$a_1 w_1 + \cdots + a_k w_k + \underbrace{a_{k+1} w_{k+1} + \cdots + a_n w_n}_{\text{at least one of these coefficients is \underline{not} zero}} = 0$$

This contradicts that $\{w_1, \dots, w_n\}$ is a basis of V.

Thus $\{T(w_{k+1}), \dots, T(w_n)\}$ is linearly independent.

So $\{T(w_{k+1}), \dots, T(w_n)\}$ is a basis of $\operatorname{Im}(T)$. Now:

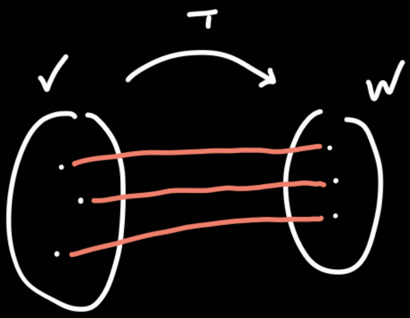
$$\dim(V) = n = k + (n-k) = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{Im}(T)). \qquad \square.$$

Def: Let $T: V \to W$ be a linear transformation.

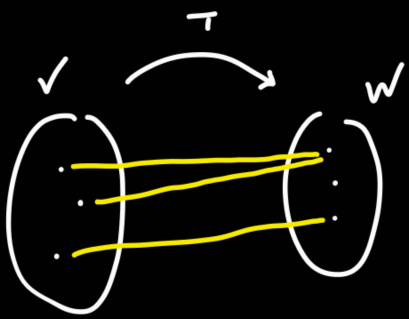
We say that T is <u>injective</u> when $T(x) = T(y)$ implies $x = y$ for all $x, y \in V$.

We say that T is <u>surjective</u> when for each $y \in W$ there is an $x \in V$ such that $T(x) = y$.

<u>Injective:</u>

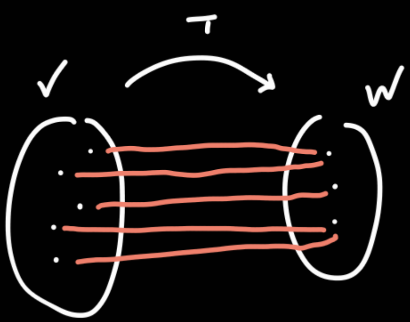


ALLOWED

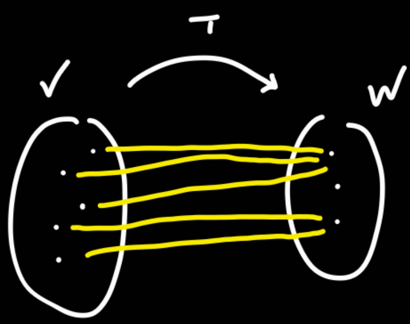


<u>NOT</u> ALLOWED

<u>Surjective:</u>



ALLOWED



<u>NOT</u> ALLOWED

Theorem: Let $T: V \to W$ be a linear transformation.

1) T is injective if and only if $\ker(T) = \{0\}$.

2) T is surjective if and only if $\operatorname{Im}(T) = W$.

Proof: 1) $\Rightarrow$) Assume T is injective. We have $\{0\} \subseteq \ker(T)$ since $\ker(T)$ is a vector subspace. Let $x \in \ker(T)$, namely $T(x) = 0$. Now:

$T(x) = 0 = T(0)$, since T is injective this means $x = 0$.

Thus $\ker(T) \subseteq \{0\}$.

$\Leftarrow$) Assume $\ker(T) = \{0\}$. We have to prove that if $T(x) = T(y)$

then $x = y$. Let $x, y \in V$ with $T(x) = T(y)$. Then:

$$T(x-y) = T(x) - T(y) = 0 \quad \text{so} \quad x - y \in \ker(T) = \{0\}.$$

Thus $x - y = 0$ so $x = y$. Thus T is injective.

2) Try it yourself! $\square$.