Theorem: Let T be a linear transformation, let V, W be finite dimensional, and

$$T: V \longrightarrow W$$

assume $\dim(V) = \dim(W)$. Then the following are equivalent:

1) T injective.

2) T surjective.

3) $\dim(\mathrm{Im}(T)) = \dim(V)$. V is the source $\mathrm{Im}(T)$ is the target

Proof: 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1)

1) $\Leftrightarrow$ 3) We know by Rank-Nullity that: $\dim(V) = \dim(\ker(T)) + \dim(\mathrm{Im}(T))$.

$$T \text{ injective} \overset{\text{Theorem of Wednesday}}{\Rightarrow} \ker(T) = \{0\} \overset{\text{The empty set } \emptyset \text{ is a basis of } \{0\}.}{\Rightarrow} \dim(\ker(T)) = 0 \overset{\text{Rank-Nullity}}{\Rightarrow} \dim(V) = \dim(\mathrm{Im}(T)).$$

$$T \text{ injective} \Leftarrow \ker(T) = \{0\} \Leftarrow \dim(\ker(T)) = 0 \Leftarrow \dim(V) = \dim(\mathrm{Im}(T)).$$

2) $\Leftrightarrow$ 3)

$$T \text{ surjective} \overset{\text{Theorem from Wednesday}}{\Leftrightarrow} \mathrm{Im}(T) = W \overset{\text{Theorem from Friday}}{\Leftrightarrow} \dim(W) = \dim(\mathrm{Im}(T)) \overset{\dim(V) = \dim(W)}{\Leftrightarrow} \dim(V) = \dim(\mathrm{Im}(T))$$

$\square$.

Moral: Often computing dimensions gives all the necessary information, and we do not need to exactly compute $\ker(T)$ or $\mathrm{Im}(T)$.

Moral: To check properties of a linear transformation we only need to check them on a basis of the source.

$\hookrightarrow$ linear, injective, surjective, ker, image, ...

In fact, to completely determine a linear transformation $T: V \to W$, it suffices to say what it does to a basis of V.

Theorem: Let $T: V \to W$, let $\{v_1, \dots, v_n\}$ be a basis of V. If $T': V \to W$ is a linear transformation such that $T(v_i) = T'(v_i)$ for all $i = 1, \dots, n$, then T and T' are the same linear transformation.

Proof: Let $v \in V$, then $v = a_1 \cdot v_1 + \dots + a_n \cdot v_n$ for some scalars $a_1, \dots, a_n \in \mathbb{F}$. Then:

$$T(v) = T(a_1 v_1 + \dots + a_n v_n) \overset{T \text{ linear}}{=} T(a_1 v_1) + \dots + T(a_n v_n) \overset{T \text{ linear}}{=}$$

$$= a_1 \cdot T(v_1) + \dots + a_n \cdot T(v_n) \overset{T(v_i) = T'(v_i)}{=} a_1 \cdot T'(v_1) + \dots + a_n \cdot T'(v_n) \overset{T' \text{ linear}}{=}$$

$$= T'(a_1 \cdot v_1) + \dots + T'(a_n \cdot v_n) \overset{T' \text{ linear}}{=} T'(a_1 v_1 + \dots + a_n v_n) = T'(v).$$

Thus $T(v) = T'(v)$ for all $v \in V$, so T and T' are the same. $\square$.

Example: Let $T: \mathbb{R}_2[x] \longrightarrow \mathbb{R}_3[x]$.

$$f(x) \longmapsto 2 \cdot f'(x) + \int_0^x 3 f(x)\, dx.$$

Question: is T linear? What is $\mathrm{Im}(T)$? What is $\ker(T)$?

T is linear because it is a combination of linear transformations (derivatives and integrals).

Take $\beta = \{1, x, x^2\}$ a basis of $\mathbb{R}_2[x]$. Now:

$$T(1) = 3x, \qquad T(x) = 2 + \frac{3x^2}{2}, \qquad T(x^2) = 4x + x^3.$$

Now: $\mathrm{Im}(T) = \mathrm{Span}\{\underbrace{3x,\ 2+\frac{3x^2}{2},\ 4x+x^3}_{\text{linearly independent}}\}$ so $\mathrm{Im}(T)$ has dimension 3.

Thus T is <u>not</u> surjective.

By Rank-Nullity: $\underbrace{\dim(V)}_{3} = \dim(\ker(T)) + \underbrace{\dim(\mathrm{Im}(T))}_{3}$.

Thus $\dim(\ker(T)) = 0$ so T is injective.

$\mathbb{R}^n \quad \mathbb{F}^n \qquad \mathbb{F}^m \xrightarrow{T} \mathbb{F}^n \qquad T \in M_{n\times m}(\mathbb{F})$

<u>Def</u>: Let V be a vector space, $\{v_1, \dots, v_n\} = \beta$ a basis of V, let $v \in V$ so we can write $v = a_1 v_1 + \dots + a_n \cdot v_n$. The <u>coordinate vector</u> of v in terms of β is:

$$[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

We should understand $[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ as just notation for $v = a_1 v_1 + \dots + a_n \cdot v_n$.

<u>Example:</u> Let $V = \mathbb{R}_2[x]$, let $p(x) = 3 - 2x + 4x^2$ be a vector in $\mathbb{R}_2[x]$.

Now $\mathbb{R}_2[x]$ has basis $\beta = \{1, x, x^2\}$ and $\gamma = \{1+x, 1-x, 3x^2\}$.

We have:

$$p(x) = 3 - 2x + 4x^2 \longleftrightarrow [p(x)]_\beta = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$p(x) = \frac{1}{2}(1+x) + \frac{5}{2}(1-x) + \frac{4}{3} \cdot 3x^2 \longleftrightarrow [p(x)]_\gamma = \begin{bmatrix} 1/2 \\ 5/2 \\ 4/3 \end{bmatrix}$$