Recall: $T: V \longrightarrow V$

Preferred directions of T: $\underset{v}{\text{eigenvectors}}$, $\underset{\lambda}{\text{eigenvalue}}$. $T(v) = \lambda \cdot v$

If $A \in M_{n\times n}(\mathbb{F})$ then $T_A: \mathbb{F}^n \longrightarrow \mathbb{F}^n$ has eigenvalue λ when $\det(\underset{\substack{\text{not} \\ \text{invertible.}}}{\underline{A-\lambda\cdot I_n}})=0$.

Definition: Let $T: V \longrightarrow V$ be a linear transformation, let V be finite dimensional, let β be a basis of V. The <u>characteristic polynomial</u> of T is:

$$p_T(x) = \det\left([T - x\cdot id_V]_\beta^\beta\right).$$

If $\dim_{\mathbb{F}}(V) = n$ then $p_T(x)$ is a polynomial of degree at most n.

$$p_T(x) = \det\left([T - x\cdot id_V]_\beta^\beta\right) = \det\left([T]_\beta^\beta - [x\cdot id_V]_\beta^\beta\right) =$$

$$= \det\left([T]_\beta^\beta - x\cdot [id_V]_\beta^\beta\right) = \det\left([T]_\beta^\beta - x\cdot I_{n\times n}\right).$$

Theorem: Let $T: V \longrightarrow V$ be a linear transformation, let V be finite dimensional.

A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if λ is a root of the characteristic polynomial $p_T(x)$ of T.

Very technical proof. **Idea:** $T(v) = \underset{\text{eigenvalue}}{\lambda}\cdot \underset{\text{eigenvector}}{v} \iff T(v) - \lambda\cdot v = 0$

$\boxed{\text{Theorem 50.}}$ $\iff (T - \lambda\cdot id_V)(v) = 0$

<u>Theorem:</u> Let $A \in M_{n\times n}(\mathbb{F})$ be an upper diagonal matrix. Then the eigenvalues of A are its diagonal entries.

the determinant of an upper triangular matrix is the multiplication of its diagonal elements.

<u>Proof:</u> $p_A(x) = \det(\underbrace{A - x\cdot I_{n\times n}}_{\substack{\text{this is upper triangular}\\ \text{when } A \text{ is upper triangular}}}) \downarrow = (A_{11}-x)(A_{22}-x)\cdots(A_{nn}-x)$.

The roots of $p_A(x)$ are $A_{11}, A_{22}, \dots, A_{nn}$. □.

<u>Recall:</u> To find eigenvalues and eigenvectors was:

1. Find eigenvalue λ using determinants.

2. Solve $T(v) = \lambda\cdot v$ to find eigenvector(s).

$$(\underbrace{T-\lambda}_{\text{linear transformation}}\cdot id_V)(v) = 0$$

$T-\lambda\cdot id_V : V \longrightarrow V$

<u>Theorem:</u> Let $T: V \longrightarrow V$ be linear, V finite dimensional. A vector $v \in V$ is an eigenvector of T with eigenvalue $\lambda \in \mathbb{F}$ if and only if v is <u>not</u> zero and $v \in \ker(T-\lambda\cdot id_V)$ for $\lambda \in \mathbb{F}$.

<u>Proof:</u> $\Rightarrow$) Suppose $v \in V$ is an eigenvector. Then $v \neq 0$ and $T(v) = \lambda\cdot v$ so
λ associated eigenvalue
$(T-\lambda\cdot id_V)(v) = 0$ so $v \in \ker(T-\lambda\cdot id_V)$.

$\Leftarrow$) Suppose $v \neq$ and $v \in \ker(T-\lambda\cdot id_V)$. Then $(T-\lambda\cdot id_V)(v) = 0$ so $T(v) = \lambda\cdot v$. □.

Recall: If v is an eigenvector of eigenvalue λ of $T: V \to V$ then any scalar multiple of v is also an eigenvector of eigenvalue λ.

$c \cdot v \qquad T(c\cdot v) = c\cdot T(v) = c\cdot\lambda\cdot v = \lambda\cdot c\cdot v = \lambda\cdot(c\cdot v).$

Note: If v and w are distinct eigenvectors with the same eigenvalue λ then :

Span(v, w) is a vector subspace of V where every single $u \in \mathrm{Span}(v,w)$ is an eigenvector of eigenvalue λ.

Definition: Let $T: V \to V$ linear, V finite dimensional, $\lambda \in \mathbb{F}$. The vector subspace :

$$\underbrace{E_\lambda} = \ker(T - \lambda\cdot \mathrm{id}_V)$$ is called the eigenspace of λ.

these are T invariant subspaces.