Self-adjoint and normal linear transformations.

Goal: When V is an inner product vector space, V has orthonormal basis. Can we have diagonalizable linear transformations whose eigenbasis is orthonormal?

Spoiler: Over $\mathbb{C}$ this is possible, and these linear transformations are exactly the normal ones.

Assume all vector spaces are inner product vector spaces and finite dimensional.

Definition: A linear transformation $T: V \to V$ is self-adjoint when $T = T^*$.

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle = \langle v, T(w) \rangle.$$

Proposition: Let $T: V \to V$ be a self-adjoint linear transformation. Then the eigenvalues of T are real numbers.

Proof: Let λ be an eigenvalue of T with v its associated eigenvector. Then:

$$\lambda \cdot \|v\|^2 = \lambda \cdot \langle v, v \rangle = \langle \lambda \cdot v, v \rangle = \langle T(v), v \rangle = \langle v, T^*(v) \rangle = \langle v, T(v) \rangle =$$

$$= \langle v, \lambda \cdot v \rangle = \bar{\lambda} \cdot \langle v, v \rangle = \bar{\lambda} \cdot \|v\|^2$$

Since $v \neq 0$ then $\|v\|^2 \neq 0$ and $\lambda = \bar{\lambda}$ so $\lambda \in \mathbb{R}$. $\square$.

Proposition: Let $T: V \to V$ be a self-adjoint linear transformation. If $\langle T(v), v \rangle = 0$ for all $v \in V$ then $T = 0$. **Hint:** Consider: $\langle T(v), v \rangle \overset{\text{given}}{=} 0 = \langle v, 0 \rangle$ ← observe this happens.

$$\langle T(v), v \rangle = \langle v, T(v) \rangle$$

Proposition: Let $T: V \to V$ be a linear transformation. Then T is self-adjoint if and only if

$\langle T(v), v\rangle \in \mathbb{R}$ for all $v \in V$.

Proof: $\Rightarrow$) Suppose T is self adjoint. Then:

$$\langle T(v), v\rangle = \langle v, T^*(v)\rangle = \langle v, T(v)\rangle = \overline{\langle T(v), v\rangle} \quad \text{so } \langle T(v), v\rangle \in \mathbb{R}.$$

$\Leftarrow$) Suppose that $\langle T(v), v\rangle \in \mathbb{R}$. Then:

$$\langle T(v), v\rangle = \overline{\langle v, T(v)\rangle} = \langle v, T(v)\rangle \qquad\qquad \langle T(v), w\rangle = \langle v, T^*(w)\rangle$$

$$\uparrow$$

$$\underbrace{\overline{\langle T(v), v\rangle}}_{\in \mathbb{R}} = \overline{\overline{\langle v, T(v)\rangle}} = \underbrace{\langle v, T(v)\rangle}_{\in \mathbb{R}}$$

$$0 = \langle T(v), v\rangle - \langle T^*(v), v\rangle = \langle (T - T^*)(v), v\rangle.$$

$\uparrow$

because $\langle v, T^*(v)\rangle = \langle T(v), v\rangle = \langle v, T(v)\rangle$ implies $\langle T(v), v\rangle = \langle T^*(v), v\rangle$.

Apply this to the vector $v + w$:

$$0 = \langle (T - T^*)(v+w), v+w\rangle \quad \text{so} \quad \langle (T-T^*)(v), w\rangle + \langle (T-T^*)(w), v\rangle = 0.$$

$$0 = \langle (T - T^*)(i\cdot v+w), i\cdot v+w\rangle \quad \text{so} \quad \langle (T-T^*)(v), w\rangle - \langle (T-T^*)(w), v\rangle = 0.$$

So: $\langle (T-T^*)(v), w\rangle = 0$ for all $v, w \in V$. Thus $T - T^* = 0$. $\square$.

Namely: self-adjoint linear transformations behave like real numbers.

Definition: A linear transformation $T: V \to V$ is <u>normal</u> when $T^* \cdot T = T \cdot T^*$.

Note that every self-adjoint operator is normal.

Proposition: Let $T: V \to V$ be a normal linear transformation. Then $\|T(v)\| = \|T^*(v)\|$ for all $v \in V$.

Proposition: Let $T: V \to V$ be a normal linear transformation. Then:

(i) $T - a \cdot id_V$ is normal for all $a \in \mathbb{F}$.

(ii) Suppose $T(v_1) = \lambda_1 \cdot v_1$ and $T(v_2) = \lambda_2 \cdot v_2$ with $v_1, v_2 \neq 0$ and $\lambda_2 \neq \lambda_1$. Then v_1 is orthogonal to v_2.

Proposition: Let $T: V \to V$ be a normal linear transformation, let $\lambda \in \mathbb{F}$ is an eigenvalue of T. Then $\bar{\lambda}$ is an eigenvalue of T^*.

Proof: Suppose $T(v) = \lambda v$ with $v \neq 0$. Then: $(T - \lambda \cdot id_V)(v) = 0$ so $\|(T - \lambda \cdot id_V)(v)\| = 0$

so $\|(T - \lambda \cdot id_V)^*(v)\| = 0$ so $\|(T^* - (\lambda \cdot id_V)^*)(v)\| = 0$ so $\|(T^* - \bar{\lambda} \cdot id_V)(v)\| = 0$

so $(T^* - \bar{\lambda} \cdot id_V)(v) = 0$ so $T^*(v) = \bar{\lambda} \cdot v$. $\square$.

Corollary: Let $T: V \to V$ be normal. Then T is diagonalizable if and only if T^* is diagonalizable.

Lemma: Let $T: V \longrightarrow V$ be a linear transformation, let β a basis of V with $[T]_\beta^\beta$ upper triangular. Then there exists an orthonormal basis γ of V with $[T]_\gamma^\gamma$ upper triangular.

Proof: Gram-Schmidt. $\square$.

Corollary: Let $\mathbb{F} = \mathbb{C}$. Let $T: V \to V$ be a linear transformation. Then there exists an orthonormal basis γ of V with $[T]_\gamma^\gamma$ is upper triangular. Exercise: 5.4.32.

Theorem: (Complex Spectral Theorem) Let $\mathbb{F} = \mathbb{C}$. A linear transformation $T: V \to V$ is diagonalizable with orthonormal eigenbasis if and only if $T: V \to V$ is normal.

Proof: $\Rightarrow$) Let β be an orthonormal eigenbasis of V. Then $[T]_\beta^\beta$ is diagonal. Then:

$[T^*]_\beta^\beta = (\overline{[T]_\beta^\beta})^t$ is diagonal. Now:

$$[T^* T]_\beta^\beta = [T^*]_\beta^\beta \cdot [T]_\beta^\beta = [T]_\beta^\beta \cdot [T^*]_\beta^\beta = [T T^*]_\beta^\beta$$

(diagonal matrices commute)

So $T^* T = T T^*$ and T is normal.

$\Leftarrow$) Suppose T normal. By the Corollary above there is an orthonormal basis $\beta = \{e_1, \dots, e_n\}$ of V with $[T]_\beta^\beta$ upper triangular.

$$[T]_\beta^\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \text{so} \quad [T^*]_\beta^\beta = \left([T]_\beta^\beta\right)^t = \begin{bmatrix} \overline{a_{11}} & 0 & 0 & \cdots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & & \cdots & \overline{a_{nn}} \end{bmatrix}.$$

↑ $[T(e_1)]_\gamma$ ↑ $[T^*(e_1)]_\gamma$

Thus: $T(e_1) = a_{11} \cdot e_1$ and $T^*(e_1) = \overline{a_{11}} \cdot e_1 + \overline{a_{12}} \cdot e_2 + \cdots + \overline{a_{1n}} \cdot e_n$.

Also: $|a_{11}|^2 = \|T(e_1)\| = \|T^*(e_1)\| = |a_{11}|^2 + |a_{12}|^2 + \cdots + |a_{1n}|^2$

So: $0 = |a_{12}|^2 + \cdots + |a_{1n}|^2$ so $a_{1j} = 0$ for all $j \neq 1$.

Also: $T(e_2) = a_{12} \cdot e_1 + a_{22} \cdot e_2 = a_{22} \cdot e_2$

$T^*(e_2) = \overline{a_{22}} \cdot e_2 + \overline{a_{23}} \cdot e_3 + \cdots + \overline{a_{2n}} \cdot e_n$

So: $|a_{22}|^2 = \|T(e_2)\| = \|T^*(e_2)\| = |a_{22}|^2 + |a_{23}|^2 + \cdots + |a_{2n}|^2$

Hence: $0 = |a_{23}|^2 + \cdots + |a_{2n}|^2$ so $a_{2j} = 0$ for all $j \neq 2$.

Repeat this process. We obtain that $a_{ij} = 0$ for $i \neq j$. Hence $[T]_\beta^\beta$ is diagonal.

Thus β is a basis of eigenvectors for both T and T^*. Since β was already orthonormal, we are done. $\square$.

<u>Corollary</u>: Let $\mathbb{F} = \mathbb{C}$. Let $T: V \to V$ be normal, let $\lambda_1, \ldots, \lambda_m$ the distinct eigenvalues of T, then: $V = \ker(T - \lambda_1 \cdot id_V) \oplus \cdots \oplus \ker(T - \lambda_m \cdot id_V)$.