0. Fields.

Linear Algebra $\longrightarrow$ linear equations and linear transformations.

$\longrightarrow$ vector spaces and linear maps

objects with structure: $+$, $\cdot$ — multiplication by scalars

functions preserving this structure.

Definition: A field $\mathbb{F}$ is a set with sum and product:

$\mathbb{R}$ is a field

$$+ : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F} \qquad\qquad \cdot : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$$
$$(a,b) \longmapsto a+b \qquad\qquad (a,b) \longmapsto a\cdot b$$

such that for all $a, b, c \in \mathbb{F}$:

Commutativity (1) $a+b = b+a$ and $a\cdot b = b\cdot a$.

Associativity (2) $(a+b)+c = a+(b+c)$ and $(a\cdot b)\cdot c = a\cdot(b\cdot c)$

Units (3) There exists $0, 1 \in \mathbb{F}$ such that $a+0=a$ and $a\cdot 1 = a$.

Inverses (4) When $a \neq 0$ there exists $-a, a^{-1} \in \mathbb{F}$ such that

$$a+(-a) = 0 \quad \text{and} \quad a\cdot a^{-1} = 1.$$

Distributivity (5) $a\cdot(b+c) = a\cdot b + a\cdot c$

The elements in $\mathbb{F}$ are called scalars.

Examples: 1. Numbers: $\mathbb{Q}$ rational numbers.

$\mathbb{R}$ real numbers.

$\mathbb{C}$ complex numbers.

2. $\mathbb{Z}$ integers are <u>not</u> a field. Product does <u>not</u> have inverses.

$\mathbb{N}$ natural numbers are <u>not</u> a field. Sum does <u>not</u> have inverses.

3. Take $\mathbb{Z}$ and declare that all the even numbers are equal, and that all odd numbers are equal.

$$\dots, -2, -1, 0, 1, 2, \dots$$

"$\mathbb{Z}$ modulo 2" $\mathbb{Z}_2$, $\mathbb{Z}/(2)$, $\mathbb{Z}/2\mathbb{Z}$

$$\mathbb{Z}_2 = \{ \underbrace{[0]}_{\text{even}}, \underbrace{[1]}_{\text{odd}} \}$$

$$0+0 \underbrace{\equiv}_{\substack{\text{equivalent} \\ (\text{modulo } 2)}} 0 \qquad 0+1 \equiv 1 \qquad 1+1 \equiv 2 \equiv 0$$

$$0 \cdot 0 \equiv 0 \qquad 0 \cdot 1 \equiv 1 \qquad 1 \cdot 1 \equiv 1$$

$$-1 \equiv 1 \qquad -1^{-1} \equiv 1$$

In fact, for $p \in \mathbb{N}$ a prime number, then: $\mathbb{Z}_p = \{ [0], [1], \dots, [p-1] \}$ is a field. For $\mathbb{Z}_p$ we declare that two integers are equal if and only if they have the same remainder when divided by p.

4. $\mathbb{C} = \mathbb{R}[i]$ wher i is a solution of $x^2+1=0$ is a field.

$$\mathbb{R}[i] = \{ a + bi \mid a, b \in \mathbb{R} \}$$

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

$$(a+bi)\cdot(c+di) = ac + adi + bci + bd\cdot(-1) =$$

$$= (ac - bd) + (ad + bc)\cdot i$$

$\mathbb{Q}[\sqrt{2}]$ is a field. $\sqrt{2}$ is the solution of $x^2 - 2 = 0$.

$$\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \}$$