1. Vector spaces.

Definition: A <u>vector space</u> V over a field $\mathbb{F}$ is a set with multiplication by scalars and addition:

$\mathbb{R}^3$

$$+ : V \times V \longrightarrow V \qquad\qquad \cdot : \mathbb{F} \times V \longrightarrow V$$
$$(x,y) \longmapsto x+y \qquad\qquad (a, x) \longmapsto a\cdot x$$

such that for all $x, y, z \in V$ and $a, b \in \mathbb{F}$ then:

Commutativity (1) $x+y = y+x$

Associativity (2) $(x+y)+z = x+(y+z)$

Unit for + (3) There exists $\vec{0} \in V$ with $x + \vec{0} = x$.

Inverses for + (4) For x, there exists $-x \in V$ with $x + (-x) = \vec{0}$.

Unit in $\mathbb{F}$ (5) $\underset{\mathbb{F}}{1}\cdot \underset{\checkmark}{x} = \underset{\checkmark}{x}$ ⚠ $1 \in \mathbb{F}$, <u>not</u> in V.

"behaves like 1"

"Associativity" (6) $(\underset{\mathbb{F}}{a\cdot b})\cdot \underset{\checkmark}{x} = a\cdot(\underset{\checkmark}{b\cdot \underset{\checkmark}{x}})$

(7) $a\cdot(\underset{\checkmark}{x} + \underset{\checkmark}{y}) = \underset{\checkmark}{a\cdot x} + \underset{\checkmark}{a\cdot y}$ Distributivity of scalar multiplication with respect to addition in V.

(8) $(\underset{\mathbb{F}}{a+b})\cdot \underset{\checkmark}{x} = \underset{\checkmark}{a\cdot x} + \underset{\checkmark}{b\cdot x}$ Distributivity of scalar multiplication with respect to sum in $\mathbb{F}$.

Examples: 1. $\mathbb{R}^n = \overbrace{\mathbb{R}\times\cdots\times\mathbb{R}}^{n}$ is a vector space over $\mathbb{R}$. (n natural number).

$$+ \qquad \underbrace{(\overset{\mathbb{R}}{r_1},\dots,\overset{\mathbb{R}}{r_n})}_{\mathbb{R}^n} + \underbrace{(\overset{\mathbb{R}}{s_1},\dots,\overset{\mathbb{R}}{s_n})}_{\mathbb{R}^n} = (r_1+s_1,\dots,r_n+s_n)$$

$$\cdot \qquad a\cdot(r_1,\dots,r_n) = (a\cdot r_1,\dots,a\cdot r_n)$$

Question: Is $\mathbb{R}^n$ as above a vector space over $\mathbb{Q}$?

Yes.

Question: Is $\mathbb{R}^n$ as above a vector space over $\mathbb{C}$?

No.

Remark: Vector spaces are <u>closed</u> under addition and multiplication by scalars. Namely if $x,y\in V$ then $x+y\in V$, and if $a\in\mathbb{F}$ then $a\cdot x\in V$.

2. Linear maps: given V and W vector spaces over $\mathbb{F}$, consider:

$\mathcal{L}(V,W)$ are essentially matrices!

$$\mathcal{L}(V,W) = \left\{ f: V\longrightarrow W \;\middle|\; \begin{array}{ll} f(x+y) = f(x)+f(y) & \text{for all } x,y\in V \\ f(a\cdot x) = a\cdot f(x) & \text{and all } a\in\mathbb{F} \end{array} \right\}.$$

is a vector space over $\mathbb{F}$ via:

$$+ \qquad \begin{array}{rcl} (f+g): V & \longrightarrow & W \\ x & \longmapsto & f(x)+g(x) \end{array} \qquad (f+g)(x) = f(x)+g(x)$$

- $(a \cdot f) : V \longrightarrow W$ $\qquad (a \cdot f)(x) = a \cdot f(x)$

$x \longmapsto a \cdot f(x)$

⚠ **Not:** $f(a \cdot x) = a \cdot f(x)$.

3. $\mathbb{F}[x]$ (polynomials with coefficients in $\mathbb{F}$) is a vector space over $\mathbb{F}$.

$$a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n$$

<u>Theorem</u>: Let V be a vector space. If $x, y, z \in V$ and $x + z = y + z$ then $x = y$.

<u>Proof</u>: We know $x + z = y + z$. We will add $-z$ to both sides to obtain $x = y$.

Since $z \in V$, there exists $-z \in V$ such that $z + (-z) = \vec{0}$. Now:

$x + z = y + z \quad \Rightarrow \quad (x+z) + (-z) = (y+z) + (-z)$

Associativity $\quad \Rightarrow \quad x + (z + (-z)) = y + (z + (-z))$

Inverses + $\quad \Rightarrow \quad x + \vec{0} = y + \vec{0}$

Identity + $\quad \Rightarrow \quad x = y$ $\qquad \square$.

<u>Corollary</u>: Let V be a vector space. The vector $\vec{0}$ is unique.

<u>Proof</u>: Suppose there is $\vec{0}' \in V$ such that $z + \vec{0}' = z$ for all $z \in V$. Now:

$\underbrace{z + \vec{0}' = z}_{\text{assumption}} \underbrace{= z + \vec{0}}_{\vec{0} \text{ is identity for } +}$, so by the Theorem above $\vec{0}' = \vec{0}$. $\qquad \square$.

Corollary: Let V be a vector space, let $x \in V$. Then $-x \in V$ is unique.