Math 115A
Linear Algebra

Discussion 10

Problem 1($\star$).

Let V be a finite dimensional inner product space, let $T \in \mathcal{L}(V)$. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

Problem 2.

Let V be a finite dimensional inner product space, let $T \in \mathcal{L}(V)$, let U be a subspace of V. Prove that U is T-invariant if and only if $U^\perp$ is T^*-invariant.

Problem 3.

Let V be a finite dimensional complex inner product space, let $T \in \mathcal{L}(V)$ be normal, let U be a subspace of V. Prove that U is T-invariant if and only if U is T^*-invariant.

Problem 4.

Let V be a finite dimensional inner product space, let $T \in \mathcal{L}(V)$ be normal. Prove that $\ker(T) = \ker(T^*)$ and $\operatorname{im}(T) = \operatorname{im}(T^*)$.

Problem 5.

Let V be a finite dimensional real inner product space, let $T \in \mathcal{L}(V)$ be normal such that its characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T. Deduce that T is self-adjoint.

Problem 6.

Let V be an inner product space, let $S, T \in \mathcal{L}(V)$ be self-adjoint. Prove that TS is self-adjoint if and only if $TS = ST$.

Problem 7($\star$).

Let V be a complex inner product space, let $T \in \mathcal{L}(V)$. Define $T_1 = (T + T^*)/2$ and $T_2 = i(T^* - T)/2$.

(a) Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.

(b) Suppose that $T = U_1 + iU_2$ for $U_1, U_2 \in \mathcal{L}(V)$ self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.

(c) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.