Math 115A
Linear Algebra

Discussion 9

Problem 1.

(a) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$ if and only if there exists a symmetric matrix A with strictly positive eigenvalues such that $\langle x, y \rangle = x^t A y$ for all $x, y \in \mathbb{R}^n$. What is A when the inner product over $\mathbb{R}^n$ is $\langle x, y \rangle = x \cdot y$, the usual dot product of the vectors x and y?

(b) Let $M \in M_{n\times n}(\mathbb{C})$, we say that M is self-adjoint when $M^* = M$. Prove that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{C}^n$ if and only if there exists a self-adjoint matrix A with strictly positive eigenvalues such that $\langle x, y \rangle = \bar{x}^t A y$ for all $x, y \in \mathbb{C}^n$.

Problem 2.

(a) Prove that $||(x_1, \dots, x_n)||_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ for $1 \leq p < \infty$ is a norm on $\mathbb{R}^n$.

(b) Is $||(x_1, \dots, x_n)||_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ for $0 < p < 1$ a norm on $\mathbb{R}^n$?

(c) Prove that $||(x_1, \dots, x_n)||_\infty = \max\{|x_1|, \dots, |x_n|\}$ is a norm on $\mathbb{R}^n$.

Problem 3($\star$).

Let V be an inner product space, let W be a finite dimensional subspace of V. Prove that if $x \notin W$ then there exists $y \in W^\perp$ with $\langle x, y \rangle \neq 0$.

Problem 4($\star$).

Let V be a finite dimensional inner product space, let W be a subspace of V. Prove that V/W is isomorphic to $W^\perp$.

Problem 5.

Let V be an inner product space, and suppose that $u, v \in V$ are orthogonal. Prove that $||u + v||^2 = ||u||^2 + ||v||^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.

Problem 6.

Let V be an inner product space over $\mathbb{F}$, let $\{v_1, \dots, v_k\}$ be an orthogonal set in V, let $a_1, \dots, a_k \in \mathbb{F}$. Prove that $|| \sum_{i=1}^k a_i v_i ||^2 = \sum_{i=1}^k |a_i|^2 ||v_i||^2$.

Problem 7.

Let V be an inner product space over $\mathbb{F}$, let $T : V \to V$ be a projection. We say that T is an *orthogonal projection* whenever $\text{im}(T)^\perp = \ker(T)$.

(a) Prove that if $T \in \mathcal{L}(V)$ is an orthogonal projection then $\ker(T)^\perp = \text{im}(T)$.

(b) Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and $||P(v)|| \leq ||v||$ for all $v \in V$, then P is an orthogonal projection.