2. Linear transformations.

<u>Def:</u> Let V and W be vector subspaces of $\mathcal{U}$. An assignment $T: V \longrightarrow W$, $v \mapsto T(v)$ is said to be a <u>linear transformation</u> when:

(1) $T(v+v') = T(v) + T(v')$ for all $v, v' \in V$,

(2) $T(c \cdot v) = c \cdot T(v)$ for all $c \in \mathbb{F}$ and all $v \in V$.

<u>Example:</u> $\mathcal{U} = \mathbb{R}^n$, $V = \mathbb{R}^m$, $m \leq n$; $W = \mathbb{R}^l$, $l \leq n$; $T \in M_{l \times m}(\mathbb{R})$.

<u>Theorem:</u> Let $T: V \longrightarrow W$ be a linear transformation. Then:

(1) $T(\vec{0}) = \vec{0}$.

(2) $T(a \cdot x + b \cdot y) = a \cdot T(x) + b \cdot T(y)$ for all $a, b \in \mathbb{F}$ and all $x, y \in V$.

(3) $T(x-y) = T(x) - T(y)$ for all $x, y \in V$.

<u>Theorem:</u> Let $T: V \longrightarrow W$ and $S: V \longrightarrow W$. Then:

$$(T+S): V \longrightarrow W, \quad v \mapsto T(v) + S(v)$$ is a linear transformation.

Let $c \in \mathbb{F}$ then:

$$(c \cdot T): V \longrightarrow W, \quad v \mapsto c \cdot T(v)$$ is a linear transformation.

Theorem: Let V, W be vector subspaces of $\mathcal{U}$, let $\mathcal{L}(V,W)$ be the collection of all linear transformations from V to W. Then $\mathcal{L}(V,W)$ is a vector space.

$$\mathcal{L}(V,W) = \{ T: V \to W \mid T \text{ is a linear transformation} \}.$$

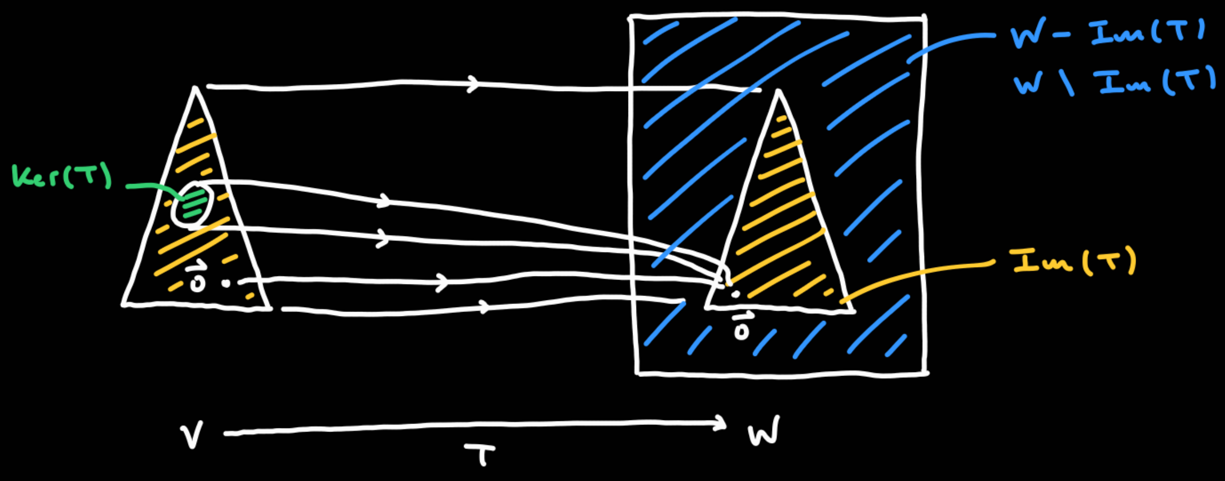
We denote $\mathcal{L}(V,V) = \mathcal{L}(V)$.

Def: Let $T: V \to W$ be a linear transformation. Consider:

kernel $\ker(T) = \{ v \in V \mid T(v) = \vec{0} \}$ also called null space.

image $\operatorname{Im}(T) = \{ w \in W \mid \text{there is a vector } v \in V \text{ with } T(v) = w \} =$

$= \{ T(v) \in W \mid v \in V \}$ also called the range.



Theorem: Let $T: V \to W$ be a linear transformation. Then $\ker(T)$ is a vector subspace of V and $\operatorname{Im}(T)$ is a vector subspace of W.

Proof: By Theorem 4 we have:

(1) Since $T(\vec{0}) = \vec{0}$ then $\vec{0} \in \ker(T)$.

(2) Let $\underbrace{x, y \in \ker(T)}_{T(x) = 0 = T(y)}$, now:

$$T(x+y) \overset{T \text{ linear}}{=} T(x) + T(y) = \vec{0} + \vec{0} = \vec{0} \quad \text{so} \quad x+y \in \ker(T).$$

(3) Let $\underbrace{x \in \ker(T)}_{T(x) = \vec{0}}$, $c \in \mathbb{F}$, now:

$$T(c \cdot x) \overset{T \text{ linear}}{=} c \cdot T(x) = c \cdot \vec{0} = \vec{0} \quad \text{so} \quad c \cdot x \in \ker(T).$$

As a consequence $\ker(T) \subseteq V$ is a vector subspace.

(1) Since $T(\vec{0}) = \vec{0}$ then $\vec{0} \in \operatorname{Im}(T)$.

(2) Let $T(x), T(y) \in \operatorname{Im}(T)$.

$$T(x) + T(y) \overset{T \text{ linear}}{=} T(x+y) \in \operatorname{Im}(T).$$

(3) Let $T(x) \in \operatorname{Im}(T)$ and $c \in \mathbb{F}$.

$$c \cdot T(x) \overset{T \text{ linear}}{=} T(c \cdot x) \in \operatorname{Im}(T).$$

Thus $\operatorname{Im}(T) \subseteq W$ is a vector subspace.

<u>Theorem:</u> Let $T: V \to W$ be a linear transformation, let $\{v_1, \dots, v_n\}$ be a basis of V.

Then $\operatorname{Im}(T) = \operatorname{Span}\{T(v_1), \dots, T(v_n)\}$. $\qquad \operatorname{Im}(T) \subseteq W$

<u>Proof:</u> $\supseteq$) Since $T(v_1), \dots, T(v_n) \in \operatorname{Im}(T)$ then $\operatorname{Span}\{T(v_1), \dots, T(v_n)\} \subseteq \operatorname{Im}(T)$.

$\subseteq$) Let $T(v) \in \operatorname{Im}(T)$ for some $v \in V$. Then: $v = a_1 v_1 + \dots + a_n v_n$. Now:

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n) \underset{\substack{\uparrow \\ T \text{ linear}}}{=} T(a_1 v_1) + \cdots + T(a_n v_n) \underset{\substack{\uparrow \\ T \text{ linear}}}{=} \underbrace{a_1 \cdot T(v_1) + \cdots + a_n \cdot T(v_n)}_{\text{in the span of } T(v_1), \ldots, T(v_n).}$$

So $T(v) \in \operatorname{Span}\{T(v_1), \ldots, T(v_n)\}$.