# Adjoints. (Section 6.3)

**Goal:** relate applying a linear transformation $T: V \to V$ inside the inner product:

$$\langle T(v), w \rangle = \langle v, S(w) \rangle \quad \text{for } S: V \to V \text{ a linear transformation.}$$

Today all vector spaces are inner product spaces and finite dimensional.

**Theorem:** Let $T: V \to \mathbb{F}$ be a linear transformation, then there exists a unique vector $u_T \in V$ such that $T(v) = \langle v, u_T \rangle$ for all $v \in V$.

In other words, every linear transformation $T: V \to \mathbb{F}$ can be realized as an inner product with a vector in $V$.

**Proof:** Let $\beta = \{e_1, \dots, e_n\}$ be an orthonormal basis of $V$. Let $v \in V$, write:

$$v = \sum_{i=1}^{n} a_i \cdot e_i = \sum_{i=1}^{n} \langle v, e_i \rangle e_i .$$

Now:

$$T(v) = T\Big(\sum_{i=1}^{n} \underbrace{\langle v, e_i \rangle}_{\in \mathbb{F}} e_i\Big) = \sum_{i=1}^{n} \underbrace{\langle v, e_i \rangle}_{\in \mathbb{F}} \cdot \underbrace{T(e_i)}_{\in \mathbb{F}} = \sum_{i=1}^{n} T(e_i) \cdot \langle v, e_i \rangle =$$

$$= \sum_{i=1}^{n} T(e_i) \cdot \overline{\langle e_i, v \rangle} = \sum_{i=1}^{n} \overline{\langle \overline{T(e_i)}\, e_i, v \rangle} = \sum_{i=1}^{n} \langle v, \overline{T(e_i)}\, e_i \rangle =$$

$$= \langle v, \sum_{i=1}^{n} \overline{T(e_i)} \cdot e_i \rangle$$

Taking $u_T = \sum_{i=1}^{n} \overline{T(e_i)} \cdot e_i$ then $T(v) = \langle v, u_T \rangle$ for all $v \in V$.

Suppose there is a vector $u_T' \in V$ with $T(v) = \langle v, u_T' \rangle$. Then:

$$0 = T(v) - T(v) = \langle v, u_T \rangle - \langle v, u_T' \rangle = \langle v, u_T - u_T' \rangle \quad \text{for all } v \in V.$$

Then $u_T - u_T' \in V^\perp = \{0\}$ so $u_T - u_T' = 0$ so $u_T = u_T'$. $\square$.

<u>Corollary</u>: Let $T: V \to W$ be a linear transformation. Then there is a unique vector

$u_W \in V$ with: $\underbrace{\langle T(v), w \rangle}_{\text{inner product in } W} = \underbrace{\langle v, u_W \rangle}_{\text{inner product in } V}$. Here $w \in W$ is fixed, and it holds for all $v \in V$.

<u>Proof</u>: Apply the Theorem above with the linear transformation:

$$\begin{aligned} \langle T(\cdot), w \rangle : V &\longrightarrow \mathbb{F} \\ v &\longmapsto \langle T(v), w \rangle \end{aligned}$$

Thus there is a vector $u_W \in V$ with $\langle T(v), w \rangle = \langle v, u_W \rangle$ for all $v \in V$. $\square$.

<u>Definition</u>: Let $T: V \to W$ be a linear transformation. We define the <u>adjoint</u> of $T$ as

the linear transformation: $\begin{aligned} T^* : W &\longrightarrow V \\ w &\longmapsto u_W \end{aligned}$ where $u_W$ is as in the Theorem

above.

Note that: $\langle T(v), w \rangle = \langle v, u_W \rangle = \langle v, T^*(w) \rangle$.

Remark that $T^*(w_1 + w_2) = T^*(w_1) + T^*(w_2)$ and $T^*(a \cdot w) = a \cdot T^*(w)$.

**Example:** Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by left multiplication with $\begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \end{bmatrix}$.

We want $T^*: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that: $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$ where the inner products are the standard ones. Then fix $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$:

$$\left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix}, T^* \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle = \left\langle T \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} y + 3z \\ 2x \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle = ya + 3za + 2xb =$$

$$= \left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} 2b \\ a \\ 3a \end{bmatrix} \right\rangle \quad \text{so} \quad T^* \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2b \\ a \\ 3a \end{bmatrix}.$$

Then $T^*: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is left multiplication by the matrix $\begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \end{bmatrix}$.

Note:

$$\overline{\begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \end{bmatrix}}^t = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \end{bmatrix}.$$

**Proposition:** Let $T, S: V \longrightarrow V$, then:

(i) $(T + S)^* = T^* + S^*$.

(ii) $(a \cdot T)^* = \bar{a} \cdot T^*$ for all $a \in \mathbb{F}$.

(iii) $(T^*)^* = T$.

(iv) $(id_V)^* = id_V$.

(v) $(ST)^* = T^* S^*$

<u>Proposition:</u> Let $T: V \to W$ linear transformation, then:

(i) $\ker(T^*) = (\mathrm{Im}(T))^\perp$.

(ii) $\mathrm{Im}(T^*) = (\ker(T))^\perp$.

(iii) $\ker(T) = (\mathrm{Im}(T^*))^\perp$.

(iv) $\mathrm{Im}(T) = (\ker(T^*))^\perp$.

<u>Theorem:</u> Let $T: V \to W$ linear transformation, $\beta$ orthonormal basis of $V$. Then:
$\gamma$ $W$

$$[T^*]_\gamma^\beta = \left(\overline{[T]_\beta^\gamma}\right)^t.$$

This is saying that the matrix associated to the adjoint $T^*$ coincides with the conjugate transpose of the matrix associated to $T$.

<u>Proof:</u> Note that we can write $T(e_i) = \sum_{j=1}^{m} \langle T(e_i), f_j \rangle \cdot f_j$. Then:

$\beta = \{e_1, \dots, e_n\}$ $\gamma = \{f_1, \dots, f_m\}$

$$[T]_\beta^\gamma = \Big[ [T(e_1)]_\gamma \cdots [T(e_n)]_\gamma \Big] = \begin{array}{c} \text{k- column} \\ \downarrow \\ \begin{bmatrix} \langle T(e_1), f_1 \rangle & \cdots & \langle T(e_n), f_1 \rangle \\ \vdots & \langle T(e_k), f_j \rangle & \vdots \\ \langle T(e_1), f_m \rangle & \cdots & \langle T(e_n), f_m \rangle \end{bmatrix} \end{array} \leftarrow \text{j-row}$$

Namely: $\left([T]_\beta^\gamma\right)_{j,k} = \langle T(e_k), f_j \rangle$.

For the same reason: $\left([T^*]_\gamma^\beta\right)_{k,j} = \langle T^*(f_j), e_k \rangle$.

Now:

$$\left(\left(\overline{[T]_\beta^\gamma}\right)^t\right)_{k,j} = \left(\overline{[T]_\beta^\gamma}\right)_{j,k} = \overline{\left([T]_\beta^\gamma\right)_{j,k}} = \overline{\langle T(e_k), f_j \rangle} =$$

$$= \overline{\langle e_k, T^*(f_j) \rangle} = \langle T^*(f_j), e_k \rangle = \left([T^*]_\gamma^\beta\right)_{k,j}.$$

Since $\left(\overline{[T]_\beta^\gamma}\right)^t$ and $[T^*]_\gamma^\beta$ are equal entry-wise, they are equal as matrices. $\square$.