MATH 115A - SPRING 2023

Pablo S. Ocal

1. Fields and vector spaces

For the non-mathematician, linear algebra is the study of linear equations and linear transformations. For us, linear algebra will be the study of linear maps between vector spaces.

We should think of vector spaces as abstract objects with special structure that behaves nicely with respect to scalars, and linear maps are functions that preserve this special structure.

Definition: A field $\mathbb{F}$ is a set with two operations

$$+ : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F} \qquad \text{and} \qquad \cdot : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$$
$$(a,b) \longmapsto a+b \qquad\qquad (a,b) \longmapsto a \cdot b$$

called sum and product respectively, such that for all $a, b, c \in \mathbb{F}$ we have:

(1) Commutativity: $a+b = b+a$ and $a \cdot b = b \cdot a$.

(2) Associativity: $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(3) Identity: there exist $0, 1 \in \mathbb{F}$ with $a+0 = a$ and $a \cdot 1 = a$.

(4) Inverses: when $a \neq 0$ there exist $-a, a^{-1} \in \mathbb{F}$ with $a+(-a) = 0$ and $a a^{-1} = 1$.

(5) Distributivity: $a \cdot (b+c) = a \cdot b + a \cdot c$.

The elements of a field are called scalars.

Example:

1. Some number sets are fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

2. Some number sets are not fields: $\mathbb{N}$, $\mathbb{Z}$. (why?)

3. There are weird fields:

$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

$\mathbb{Z}_2 = \{[0], [1]\}$ is the field of integers mod 2, having:

$$0+1=1 \quad , \quad 0+0=0 \quad , \quad 1+1=0 \quad , \quad 0 \cdot 1 = 0 \quad , \quad 1 \cdot 1 = 1.$$

We can think of $\mathbb{Z}_2$ as $\mathbb{Z}$ where we have declared that all even numbers are the same, and also that all odd numbers are the same:

$$2k \equiv 0 \quad \text{and} \quad 2k+1 \equiv 0 \quad \text{for all } k \in \mathbb{Z}.$$

$\mathbb{Z}_p = \{[0], [1], \cdots, [p-1]\}$ for $p \in \mathbb{N}$ prime is the field of integers mod p.

We can think of $\mathbb{Z}_p$ as $\mathbb{Z}$ where we declare that two numbers are equal if and only if they have the same remainder when divided by p:

$$p \cdot k + j \equiv j \text{ for all } 0 \leq j < p \text{ and all } k \in \mathbb{Z}, \text{ so}$$

$$[j] = \{\text{integers with remainder } j \text{ upon division by } p\}.$$

Definition: A vector space V over a field $\mathbb{F}$ is a set with two operations:

$+ : V \times V \longrightarrow V$ and $\cdot : \mathbb{F} \times V \longrightarrow V$

$(x,y) \longmapsto x+y$ $\quad$ $(a,x) \longmapsto a \cdot x$

called addition and scalar multiplication respectively, such that for all $x,y,z \in V$ and $a,b \in \mathbb{F}$:

(1) Commutativity of addition: $x+y = y+x$.

(2) Associativity of addition: $(x+y)+z = x+(y+z)$.

(3) Identity in V: there exists $\vec{0} \in V$ with $x+\vec{0} = \vec{0}$.

(4) Inverses in V: there exists $-x \in V$ with $x+(-x) = \vec{0}$.

(5) Scalar identity: $1 \cdot x = x$. (do we have multiplicative inverses x^{-1} in V?)

(6) Associativity of scalar multiplication: $a \cdot (b \cdot x) = (a \cdot b) \cdot x$.

(7) Distributivity of scalar multiplication over addition: $a \cdot (x+y) = a \cdot x + a \cdot y$.

(8) Distributivity of the sum over scalar multiplication: $(a+b) \cdot x = a \cdot x + b \cdot x$.

These properties are saying that the addition and multiplication by scalars in V behave well with respect to the sum and product in $\mathbb{F}$.

Remark: Alternatively, we could say that a vector space is a commutative group under addition with associative and distributive scalar multiplication.

In particular, vector spaces are closed under finite sums and scalar multiplication: if $x_1, \dots, x_n \in V$ and $a_1, \dots, a_n \in \mathbb{F}$, then $a_1 x_1 + \cdots + a_n x_n \in V$.

When $\mathbb{F} = \mathbb{R}$, we say that V is a real vector space. When $\mathbb{F} = \mathbb{C}$ we say that V is a complex vector space.

Examples:

1. $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ is called the real n-space.

   The elements in $\mathbb{R}^n$ are n-tuples $(r_1, \ldots, r_n)$ with $r_1, \ldots, r_n \in \mathbb{R}$.

   The vector addition is done componentwise:

   $$(r_1, \ldots, r_n) + (s_1, \ldots, s_n) = (r_1 + s_1, \ldots, r_n + s_n)$$

   The scalar multiplication is done componentwise:

   $$a \cdot (r_1, \ldots, r_n) = (a \cdot r_1, \ldots, a \cdot r_n)$$

   Remark: Here we could replace the field $\mathbb{R}$ by $\mathbb{Q}$, and everything would still make sense. It is important to specify over which field we are working.

   In fact, if we replace $\mathbb{R}$ by $\mathbb{Z}$, things still make sense. When we work over a ring instead of a field, we generalize vector spaces to the notion of modules.

2. Let $\mathbb{F}$ be a field, let S be a set, let V be the set of functions from S to $\mathbb{F}$.

   Namely elements $f \in V$ are functions of sets $f : S \to \mathbb{F}$.

   The scalar multiplication $a \cdot f$ is the function satisfying $(a \cdot f)(x) = a \cdot f(x)$.

The addition $f+g$ is the function satisfying $(f+g)(x) = f(x) + g(x)$.

$$a \cdot f : S \longrightarrow \mathbb{F} \qquad\qquad f+g : S \longrightarrow \mathbb{F}$$
$$x \longmapsto a \cdot f(x) \qquad\qquad x \longmapsto f(x) + g(x)$$

Many important examples arise in this way.

2.1. Let V be the set of continuous functions over $\mathbb{R}$ or over $\mathbb{C}$, denoted $\mathcal{C}(\mathbb{R})$ or $\mathcal{C}(\mathbb{C})$.

2.2. Let V be the set of polynomials with coefficients in $\mathbb{F}$, denoted $\mathbb{F}[x]$. Recall that $p(x) \in \mathbb{F}[x]$ has the form $p(x) = a_n x^n + \cdots + a_1 x + a_0$ for $a_n, \ldots, a_0 \in \mathbb{F}$.

2.3. Let V be the set of symmetric polynomials in n-variables, denoted $\mathrm{Sym}_n(\mathbb{F})$.

The elements are polynomials in the variables $x_1, \ldots, x_n$ such that:

$$p(x_1, \ldots, x_i, \ldots, x_j \ldots, x_n) = p(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n) \quad \text{for all } i,j \in \{1, \ldots, n\}.$$

That is, exchanging two variables does not change the polynomial.

Fix $n=3$, then:

$p(x_1, x_2, x_3) = x_1 + x_2 + x_3$ is symmetric,

$q(x_1, x_2, x_3) = x_1 + x_2$ is not symmetric since $q(x_1, x_3, x_2) = x_1 + x_3 \neq q(x_1, x_2, x_3)$.

$r(x_1, x_2, x_3) = x_1 x_2 + 2x_1 x_3 + x_2 x_3$ is not symmetric,

$S(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$ is symmetric.

3. Let $\mathbb{F}$ be a field, let V be the set of $n \times m$ matrices with entries in $\mathbb{F}$, denoted $M_{n\times m}(\mathbb{F})$. The matrix addition and scalar multiplication are both defined componentwise.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{bmatrix}$$

$$a \cdot \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a \cdot a_{11} & \cdots & a \cdot a_{1m} \\ \vdots & & \vdots \\ a \cdot a_{n1} & \cdots & a \cdot a_{nm} \end{bmatrix}$$

The zero vector is the zero matrix.

$$\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

In general, $M_{n\times m}(\mathbb{F})$ is not a field since we cannot multiply two $n \times m$ matrices.

4. Let V be the field of rational functions over $\mathbb{F}$, denoted $\mathbb{F}(x)$. Elements in $\mathbb{F}(x)$ are fractions of polynomials, namely $\frac{p(x)}{q(x)}$ with $p(x), q(x) \in \mathbb{F}[x]$. Now $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$, and $\mathbb{F}[x]$ is also a field on its own.

The vector addition is:

$$\frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x) + q(x)r(x)}{q(x)s(x)}$$

$$\frac{p(x)}{q(x)} + \frac{s(x)}{s(x)} = \frac{\quad}{q(x)\,s(x)}.$$

The scalar multiplication is:

$$a \cdot \frac{p(x)}{q(x)} = \frac{a \cdot p(x)}{q(x)}.$$

With these two operations, $\mathbb{F}(x)$ is a vector space over $\mathbb{F}$. Consider the sum:

$$\frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)\,s(x) + q(x)\,r(x)}{q(x)\,s(x)}$$

and the product:

$$\frac{p(x)}{q(x)} \cdot \frac{r(x)}{s(x)} = \frac{p(x)\,r(x)}{q(x)\,s(x)}.$$

With these two operations, $\mathbb{F}(x)$ is a field. The identities of $\mathbb{F}(x)$ are:

$z(x) = 0$ the zero, and $e(x) = 1$ the one.

Note that $\mathbb{F}(x)$ is closed since the sum and product give rational functions.

In fact, $\mathbb{F}(x)$ is also a vector space over $\mathbb{F}(x)$. The difference is that here the scalars have changed from $\mathbb{F}$ to $\mathbb{F}(x)$.

* Small aside on proof techniques:

We will mainly be using four techniques:

1. Induction: this is useful when proving something for all natural numbers.

Example: Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

This is true for $n=1$ since $1 = \frac{1\cdot(1+1)}{2}$.

Suppose this is true for n. We now prove it for n:

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n\cdot(n+1)}{2} + (n+1) = \frac{n\cdot(n+1)+2\cdot(n+1)}{2} =$$

$$= \frac{(n+1)\cdot(n+2)}{2} = \frac{(n+1)\cdot((n+1)+1)}{2}. \qquad \square.$$

2. Using the definition: this is useful when we do not know much.

Example: Prove that $\mathbb{Z}_2$ is a field. We only have to check that all the axioms hold true.

3. Using theorems and other results: this is useful when we know a lot.

Example: Prove that every $p(x) \in \mathbb{C}[x]$ factors into linear terms.

Let n be the degree of $p(x)$. If $n = 0, 1$ we are done. If $n \neq 0, 1$, by the Fundamental Theorem of Algebra $p(x)$ has one root $a_1 \in \mathbb{C}$. Then $x - a_1$ divides $p(x)$ so $p(x) = (x-a_1)\cdot q(x)$ with $q(x)$ of degree $n-1$. If $n-1 = 1$ then $q(x) = x - a_2$ so $p(x) = (x-a_1)(x-a_2)$ and we are done. If $n-1 \neq 1$, apply the Fundamental Theorem of Algebra again. Since at every step we are lowering the degree by 1, we will repeat this exactly n times, so we will have $p(x) = (x-a_1)\cdots(x-a_n)$. Thus $p(x)$ factors

into linear terms, as desired. □.

4. Follow your nose: when we are given several hypothesis and we are asked to verify that a statement holds, often we have to "put all the hypothesis in a box, shake it up a bit, and our desired conclusion will fall out."

Example: Prove that $\sqrt{2}$ is not a rational number.

Suppose that $\sqrt{2}$ is rational. We could then write $\sqrt{2} = \frac{a}{b}$ for $a, b \in \mathbb{Z}$.

Then squaring this we obtain $2 = \frac{a^2}{b^2}$ so $2b^2 = a^2$. Thus a^2 is even.

Since even times even is even, and odd times odd is odd, a is even.

Hence there exists $k \in \mathbb{Z}$ with $a = 2k$, so $2b^2 = (2k)^2 = 4k^2$. This means that $b^2 = 2k^2$, so as before b is even. All in all, we proved that if $\sqrt{2} = \frac{a}{b}$ then a and b are both even. However, rational numbers can be written in an irreducible way, that is, if $\sqrt{2} \in \mathbb{Q}$ then there are $p, q \in \mathbb{Z}$ such that p and q do not share any divisors and $\sqrt{2} = \frac{p}{q}$. This is a contradiction with what we just proved: p and q both should be divisible by 2. Thus $\sqrt{2} \notin \mathbb{Q}$. □.

End of the aside. *

We now prove some properties of vector spaces.

Theorem 1: Let V be a vector space. If $x, y, z \in V$ and $x+z = y+z$ then $x = y$.

Proof: Since $z \in V$, by axiom 4 there is $-z \in V$ with $z + (-z) = 0$. Hence:

$$x+z = y+z \Rightarrow (x+z)+(-z) = (y+z)+(-z)$$

Associativity (2) $\Rightarrow$ $x+(z+(-z)) = y+(z+(-z))$

Inverses (4) $\Rightarrow$ $x+0 = y+0$

Identity (3) $\Rightarrow$ $x = y$. $\square$.

Corollary 2: Let V be a vector space. The vector $\vec{0} \in V$ is unique.

Proof: Suppose that there is a vector $\vec{0}' \in V$ such that $z + \vec{0}' = z$ for all $z \in V$.

Now $z + \vec{0} = z = z + \vec{0}'$ so by Theorem 1 we have $\vec{0} = \vec{0}'$. Every vector in V that satisfies axiom 3 is equal to $\vec{0}$, so $\vec{0}$ is unique. $\square$.

Corollary 3: Let V be a vector space, fix $x \in V$. Then $-x \in V$ is unique.

Proof: Analogous to the above.

Given a mathematical object with structure, we always look at how that structure reappears in mathematical objects inside the original one. For sets, these are subsets. For

vector spaces, we look at vector subspaces.

Definition: Let V be a vector space over $\mathbb{F}$. A vector subspace W of V is a subset of V that is also a vector space with the addition and multiplication by scalars inherited from V.

Example:

1. Over $\mathbb{Q}$ we have $\mathbb{Q}^n \subsetneq \mathbb{R}^n \subsetneq \mathbb{C}^n$ are all subspaces of $\mathbb{C}^n$.
2. Over $\mathbb{Q}$ we have $\mathbb{Q}[x] \subsetneq \mathbb{R}[x] \subsetneq \mathbb{C}[x]$ are all subspaces of $\mathbb{C}[x]$.
3. Over $\mathbb{Q}$ we have $M_{n\times m}(\mathbb{Q}) \subsetneq M_{n\times m}(\mathbb{R}) \subsetneq M_{n\times m}(\mathbb{C})$ are all subspaces of $M_{n\times m}(\mathbb{C})$.
4. The symmetric polynomials in n variables $\mathrm{Sym}_n(\mathbb{F})$ are a subspace of the vector space of all the polynomials in n variables $\mathbb{F}[x_1, \cdots, x_n]$.

Theorem 4: Let V be a vector space. A subset W of V is a subspace of V if and only if all the following hold:

(1) $\vec{0} \in W$.

(2) $x+y \in W$ for all $x, y \in W$.

(3) $a \cdot x \in W$ for all $a \in \mathbb{F}$.

Proof: ($\Rightarrow$) Suppose that W is a subspace of V. We want to show that (1), (2), (3) hold. Since W is a vector space, there is $\vec{0}' \in W$ such that $w + \vec{0}' = w$ for all $w \in W$. Since W is a subset of V then $w \in V$ so $w + \vec{0} = w$. Now by Theorem 1 we have $\vec{0}' = \vec{0}$, so (1) holds. Since W is a vector space, (2) and (3) hold by definition.

($\Leftarrow$) Suppose that (1), (2), (3) hold. We have to verify that the following hold.

1. Commutativity of addition: it already is, $+$ is closed by (2).

2. Associativity of addition: it already is.

3. Identity in W: $\vec{0} \in W$ by (1).

4. Inverses in W: let $w \in W$, now $(-1) \cdot w \in W$ by (3), and $-w = (-1) \cdot w$. why? Prove it!

5. Scalar identity: let $w \in W$, since $w \in V$ then $1 \cdot w = w$, and by (3) we have $1 \cdot w \in W$, so $1 \cdot w = w$ is an equality in W.

6. Associativity of scalar multiplication: it already is.

7. Distributivity of scalar multiplication over addition: it already is.

8. Distributivity of the sum over scalar multiplication: it already is. $\square$.

This emphasizes that W needs to be a subset of V that is closed under the same addition

and multiplication by scalars as V.

A particularly important vector space is $M_{n\times m}(\mathbb{F})$. We now recall some definitions:

1. Column vectors: $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$ with $a_1, \dots, a_n \in \mathbb{F}$.

2. Row vectors: $(a_1, \dots, a_n) \in \mathbb{F}^n$ with $a_1, \dots, a_n \in \mathbb{F}$.

3. Square matrices: $M_{n\times n}(\mathbb{F})$, also denoted $M_n(\mathbb{F})$.

4. Diagonal matrices: $\begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} \in M_{n\times n}(\mathbb{F})$ with $a_{ij} = 0$ for $i \neq j$.

5. Upper triangular matrices: $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn} \end{bmatrix} \in M_{n\times n}(\mathbb{F})$ with $a_{ij} = 0$ for $i > j$.

   Similarly, we have lower triangular matrices.

5.1. Strictly upper triangular matrices: we instead require $a_{ij} = 0$ for $i \geq j$, so the diagonal entries are also zero.

   Similarly, we have strictly lower triangular matrices.

6. Zero matrix: $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in M_{n\times m}(\mathbb{F})$ with $a_{ij} = 0$ for all i, j.

7. Identity matrix: $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in M_{n\times n}(\mathbb{F})$ the diagonal matrix with $a_{ii} = 1$ for all i, and $a_{ij} = 0$ for $i \neq j$.

8. Transpose of a matrix: Let $M = \begin{bmatrix} a_{11} & \cdots & a_{1m} \end{bmatrix} \in M_{n\times m}(\mathbb{F})$, its transpose is

8. Transpose of a matrix: let $M = \begin{bmatrix} \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in M_{mxn}(\mathbb{F})$ its <u>transpose</u> is $M^t = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \in M_{nxm}(\mathbb{F})$ the matrix with a_{ji} in the i-th row and j-th column.

9. Trace of a matrix: let $M = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in M_{nxn}(\mathbb{F})$ its <u>trace</u> is:

$$tr(M) = \sum_{i=1}^{n} a_{ii}, \quad \text{the sum of the diagonal entries.}$$

<u>Examples:</u>

1. Symmetric matrices are a subspace of square matrices.

Recall that a matrix is symmetric when it is equal to its transpose.

<u>Proof:</u> We show that $\{A \in M_n(\mathbb{F}) \mid A^t = A\}$ is a subspace of $M_n(\mathbb{F})$.

(1) $0 = 0^t$ so 0 is symmetric.

(2) If A, B are symmetric then:

$$(A+B)^t = A^t + B^t = A + B \quad \text{so } A+B \text{ is symmetric.}$$

(3) If A is symmetric and $c \in \mathbb{F}$ then:

$$(cA)^t = c(A^t) = cA \quad \text{so } cA \text{ is symmetric.} \qquad \square.$$

2. Diagonal matrices are a subspace of square matrices.

3. Upper triangular matrices are a subspace of square matrices.

4. The set of the zero matrix $\{0\}$ is a subspace of $M_{n\times m}(\mathbb{F})$. We call it the zero subspace, or the trivial subspace, of $M_{n\times m}(\mathbb{F})$.

5. The set of traceless matrices, namely matrices whose trace is zero, is a subspace of square matrices.

6. The set of polynomials with degree less than or equal to n, denoted $P_n(\mathbb{F})$ or $\mathbb{F}_n[x]$, is a subspace of the vector space of polynomials $\mathbb{F}[x]$. Here we note that the zero polynomial has any degree we want.

7. We have the inclusions of subspaces:

$$\mathbb{R} \subseteq \mathbb{R}_n[x] \subseteq \mathbb{R}[x] \subseteq \mathcal{C}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$$

Now that we know many examples of vector spaces, we would like to construct new vector subspaces from old ones.

<u>Theorem 5</u>: Let V be a vector space, let U and W be subspaces of V. Then $U \cap W$ is a subspace of V.

<u>Proof</u>: We show that $U \cap W$ satisfies conditions (1), (2), (3) of Theorem 4.

(1) We want to show $\vec{0} \in U \cap W$. Since U and W are subspaces of V, then $\vec{0} \in U$ and $\vec{0} \in W$, so $\vec{0} \in U \cap W$.

(2) We want to show $x+y \in U \cap W$ when $x, y \in U \cap W$. Suppose $x, y \in U \cap W$, then $x, y \in U$ and $x, y \in W$. Since U and W are subspaces then $x+y \in U$ and $x+y \in W$, so $x+y \in U \cap W$.

(3) We want to show $c \cdot x \in U \cap W$ when $x \in U \cap W$ and $c \in \mathbb{F}$. Suppose $x \in U \cap W$ and $c \in \mathbb{F}$. Then $x \in U$ and $x \in W$. Since U and W are subspaces then $c \cdot x \in U$ and $c \cdot x \in W$, so $c \cdot x \in U \cap W$. □.

However, $U \cup W$ is almost never a subspace of V. The correct way of "putting two vector subspaces together" is by addition.

Definition: Let V be a vector space, let U and W be subspaces of V. The internal sum or sum of U and W, denoted $U+W$, is:

$$U+W = \{u+w \mid u \in U, w \in W\}.$$

Theorem 6: Let V be a vector space, let U and W be subspaces of V. Then $U+W$ is a vector subspace of V.

Proof: We show that $U+W$ satisfies conditions (1), (2), (3) of Theorem 4.

(1) Since $\vec{0} \in U$ and $\vec{0} \in W$, then $\vec{0} = \vec{0} + \vec{0} \in U+W$.

(2) Given $u+w, u'+w' \in U+W$ then:

$$(u+w)+(u'+w') = (u+u')+(w+w') \in U+W$$

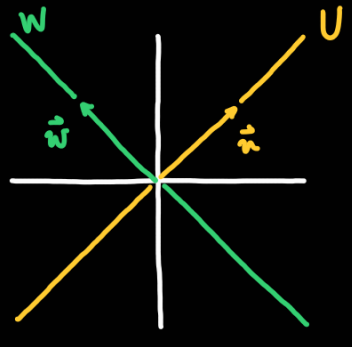
since $u+u' \in U$ and $w+w' \in W$.

(3) Given $u+w \in U+W$ and $c \in \mathbb{F}$ then:

$$c\cdot(u+w) = c\cdot u + c\cdot w \in U+W$$

since $c\cdot u \in U$ and $c\cdot w \in W$. $\square$.

Example: Let $V=\mathbb{R}^2$, let $\vec{u},\vec{w} \in V$ be non-zero and non-parallel. Set $U=\{r\cdot\vec{u} \mid r\in\mathbb{R}\}$ and $W=\{r\cdot\vec{w} \mid r\in\mathbb{R}\}$, these are vector subspaces of V. Now $U+W=V$.



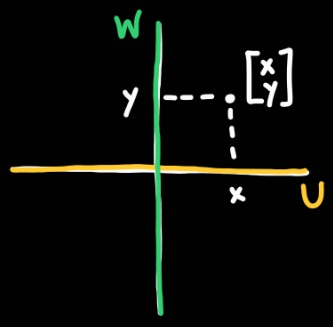
Definition: Let V be a vector space, let U and W be subspaces of V. We say that V is the direct sum of U and W, denoted $V=U\oplus W$, when the following hold.

(1) $V=U+W$.

(2) $U\cap W=\{\vec{0}\}$.

Example:

1. Let $V=\mathbb{R}^2$, let $U=\left\{\begin{bmatrix}x\\0\end{bmatrix} \mid x\in\mathbb{R}\right\}$, let $W=\left\{\begin{bmatrix}0\\y\end{bmatrix} \mid y\in\mathbb{R}\right\}$. Then $V=U\oplus W$.



2. We have: $M_n(\mathbb{F}) = \mathcal{U}_n(\mathbb{F}) \oplus \mathcal{D}_n(\mathbb{F}) \oplus \mathcal{L}_n(\mathbb{F})$ where:

$\mathcal{U}_n(\mathbb{F})$ are the strictly upper triangular matrices,

$\mathcal{D}_n(\mathbb{F})$ are the diagonal matrices,

$\mathcal{L}_n(\mathbb{F})$ are the strictly lower triangular matrices,

because any square matrix $A \in M_n(\mathbb{F})$ can be decomposed as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\,n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix} = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1\,n-1} & a_{1n} \\ 0 & 0 & \cdots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & 0 & 0 \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & 0 \end{bmatrix}.$$

There is also a way of constructing a vector subspace from a collection of vectors.

<u>Definition:</u> Let V be a vector space. A vector $v \in V$ is said to be a <u>linear combination</u> of

the vectors $v_1, \dots, v_n \in V$ if there exist scalars $a_1, \dots, a_n \in \mathbb{F}$ such that:

$$v = a_1 v_1 + \cdots + a_n v_n.$$

The scalars $a_1, \dots, a_n \in \mathbb{F}$ are also called <u>coefficients</u>.

<u>Example:</u> Let $V = \mathbb{Q}[x]$. Since: $3x^5 + 2x - 1 = (5x^6 + 3x^5 - 2) + 2 \cdot (x^6 + x) - (7x^6 + 1)$

then $v = 3x^5 + 2x - 1$ is a linear combination of $v_1 = 5x^6 + 3x^5 - 2$, $v_2 = x^6 + x$,

$v_3 = 7x^6 - 1$ with coefficients $a_1 = 1$, $a_2 = 2$, $a_3 = -1$.

Also: $3x^5 + 2x + 1 = 3 \cdot (x^5) + 2 \cdot (x) - (1)$ so $v = 3x^5 + 2x - 1$ is also a linear

combination of $w_1 = x^5$, $w_2 = x$, $w_3 = 1$ with coefficients $b_1 = 3$, $b_2 = 2$, $a_3 = -1$.

<u>Definition:</u> Let V be a vector space, let $\{v_1, v_2, \dots\} \subseteq V$ be a (possibly infinite) subset of V.

The <u>span</u> of this subset, denoted $\text{Span}\{v_1, v_2, \dots\}$, is the set of all linear combinations

of $\{v_1, v_2, \dots\}$.

$$\text{Span}\{v_1, v_2, \dots\} = \{ a_{i_1} \cdot v_{i_1} + \cdots + a_{i_n} v_{i_n} \mid a_{i_1}, \dots, a_{i_n} \in \mathbb{F} \}.$$

<u>Theorem 7:</u> Let V be a vector space, let $\{v_1, v_2, \dots\} \subseteq V$ be a subset of V. Then

$\text{Span}\{v_1, v_2, \dots\}$ is a vector subspace of V.

<u>Proof:</u> We show that $\text{Span}\{v_1, v_2, \dots\}$ satisfies conditions (1), (2), (3) of Theorem 4.

(1) Note: $\vec{0} = 0 \cdot v_1 \in \text{Span}\{v_1, v_2, \dots\}$.

(2) Given $u, v \in \text{Span}\{v_1, v_2, \dots\}$ then there exist scalars $a_{i_1}, \dots, a_{i_n}, b_{j_1}, \dots, b_{j_m} \in \mathbb{F}$ such that $u = \sum_{k=1}^{n} a_{i_k} v_{i_k}$ and $v = \sum_{k=1}^{m} b_{j_k} \cdot v_{j_k}$. Suppose that the sets $\{v_{i_1}, \dots, v_{i_n}\}$ and $\{v_{j_1}, \dots, v_{j_m}\}$ have common elements, without loss of generality we may assume that $v_{i_1} = v_{j_1}, \dots, v_{i_\ell} = v_{j_\ell}$ for some ℓ.

Then:

$$u+v = \sum_{k=1}^{n} a_{i_k} v_{i_k} + \sum_{k=1}^{m} b_{j_k} \cdot v_{j_k} =$$

$$= \sum_{k=1}^{\ell} a_{i_k} v_{i_k} + \sum_{k=\ell+1}^{n} a_{i_k} v_{i_k} + \sum_{k=1}^{\ell} b_{j_k} v_{j_k} + \sum_{k=\ell+1}^{m} b_{j_k} v_{j_k} =$$

$$= \sum_{k=1}^{\ell} (a_{i_k} + b_{j_k}) \cdot v_{i_k} + \sum_{k=\ell+1}^{n} a_{i_k} v_{i_k} + \sum_{k=\ell+1}^{m} b_{j_k} v_{j_k}$$

is a linear combination of $v_{i_1}, \dots, v_{i_n}, v_{j_{\ell+1}}, \dots, v_{j_m}$, so $u+v \in \text{Span}\{v_1, v_2, \dots\}$.

(3) Given $u \in \text{Span}\{v_1, v_2, \dots\}$ and $c \in \mathbb{F}$ then there exist scalars $a_{i_1}, \dots, a_{i_n} \in \mathbb{F}$ such that $u = \sum_{k=1}^{n} a_{i_k} v_{i_k}$. Then:

$$c \cdot u = c \cdot \left(\sum_{k=1}^{n} a_{i_k} v_{i_k} \right) = \sum_{k=1}^{n} c \cdot (a_{i_k} \cdot v_{i_k}) = \sum_{k=1}^{n} (c \cdot a_{i_k}) \cdot v_{i_k}$$

is a linear combination of $v_{i_1}, \dots, v_{i_n}$, so $c \cdot u \in \text{Span}\{v_1, v_2, \dots\}$. □.

<u>Definition:</u> Let V be a vector space, let $\{v_1, v_2, \dots\} \subseteq V$ be a (possibly infinite) subset of V.

When $V = \text{Span}\{v_1, v_2, \dots\}$ we say that $\{v_1, v_2, \dots\}$ <u>generate</u> or <u>span</u> V. We call the

elements in $\{v_1, v_2, \dots\}$ the <u>generators</u> of V.

<u>Example:</u> Let $V = \mathbb{R}^3$ and $v_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $v_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$. Since any vector $\begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$ can be expressed as:

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = \tfrac{1}{2}(a+b-c)\begin{bmatrix}1\\1\\0\end{bmatrix} + \tfrac{1}{2}(a-b+c)\begin{bmatrix}1\\0\\1\end{bmatrix} + \tfrac{1}{2}(-a+b+c)\begin{bmatrix}0\\1\\1\end{bmatrix}$$

then $\mathbb{R}^3 = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$.

<u>Definition:</u> Let V be a vector space, let $v_1, \dots, v_n \in V$. We say that $v_1, \dots, v_n$ are <u>linearly dependent</u> if there exist scalars $a_1, \dots, a_n \in \mathbb{F}$, at least one of them non-zero, such that $a_1 v_1 + \cdots + a_n v_n = \vec{0}$.

<u>Example:</u> Let $V = \mathbb{R}^3$ and $v_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\3\\0\end{bmatrix}$, $v_3 = \begin{bmatrix}0\\\frac{1}{2}\\0\end{bmatrix}$. Now:

$$v_2 - 2v_1 = \begin{bmatrix}2\\3\\0\end{bmatrix} - 2\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} = 2v_3 \quad \text{so} \quad v_3 = \tfrac{1}{2}(v_2 - 2v_1) = \tfrac{1}{2}v_2 - v_1$$

and thus $-v_1 + \frac{1}{2}v_2 - v_3 = \vec{0}$, so v_1, v_2, v_3 are linearly dependent.

<u>Definition:</u> Let V be a vector space, let $v_1, \dots, v_n \in V$. We say that $v_1, \dots, v_n$ are <u>linearly independent</u> if they are not linearly dependent.

<u>Remark:</u> The vectors $v_1, \dots, v_n$ are linearly independent if and only if when a linear combination $a_1 v_1 + \cdots + a_n v_n = \vec{0}$ in V then $a_1 = \cdots = a_n = 0$ in $\mathbb{F}$.

<u>Example:</u> Let V be a vector space, let $v \in V$, let $a \in \mathbb{F}$.

1. The empty set is linearly independent.

2. A single vector $\{v\}$ is linearly independent if and only if $v \neq \vec{0}$.

3. A vector and a scalar multiple of that vector $\{v, av\}$ are linearly dependent.

Theorem 8: Let $v_1, \ldots, v_n \in \mathbb{R}^m$. The vectors $v_1, \ldots, v_n$ are linearly independent if and only if the matrix $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ can be row reduced to the identity matrix.

Theorem 9: Let V be a vector space, let $\{v_1, \ldots, v_n\} \subset V$ be a linearly independent subset, let $v_{n+1} \in V$. The set $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly independent if and only if $v_{n+1} \notin \text{Span}\{v_1, \ldots, v_n\}$.

Proof: ($\Rightarrow$) Suppose that $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly independent. Additionally, assume that $v_{n+1} \in \text{Span}\{v_1, \ldots, v_n\}$, we want to achieve a contradiction. Since $v_{n+1} \in \text{Span}\{v_1, \ldots, v_n\}$ then there are scalars $a_1, \ldots, a_n \in \mathbb{F}$, at least one of them not zero, such that $v_{n+1} = a_1 v_1 + \cdots + a_n v_n$. Thus $a_1 v_1 + \cdots + a_n v_n + (-1) \cdot v_{n+1} = \vec{0}$ is a linear combination of elements in $\{v_1, \ldots, v_{n+1}\}$ with at least one of the coefficients non-zero. This means that $\{v_1, \ldots, v_{n+1}\}$ is linearly dependent, a contradiction. Since we reached a contradiction, our additional assumption of

$v_{n+1} \in \text{Span}\{v_1, \ldots, v_n\}$ is false, and thus $v_{n+1} \notin \text{Span}\{v_1, \ldots, v_n\}$.

($\Leftarrow$) Suppose that $v_{n+1} \notin \text{Span}\{v_1, \ldots, v_n\}$. Additionally, assume that $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly dependent, we want to achieve a contradiction. Since $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly dependent, then there are scalars $a_1, \ldots, a_{n+1} \in \mathbb{F}$, at least one of them not zero, such that $a_1 v_1 + \cdots + a_n v_n + a_{n+1} v_{n+1} = \vec{0}$. If $a_{n+1} = 0$ then $a_1 v_1 + \cdots + a_n v_n = \vec{0}$ is a linear combination of elements in $\{v_1, \ldots, v_n\}$ with at least one non-zero coefficient, meaning that $\{v_1, \ldots, v_n\}$ is linearly dependent. However, $\{v_1, \ldots, v_n\}$ is linearly independent by hypothesis, yielding a contradiction. Thus $a_{n+1} = 0$ is not possible. If $a_{n+1} \neq 0$ then it has a multiplicative inverse $\frac{1}{a_{n+1}}$ in $\mathbb{F}$, so we can rewrite $a_1 v_1 + \cdots + a_n v_n + a_{n+1} v_{n+1} = \vec{0}$ as $a_{n+1} v_{n+1} = -a_1 v_1 - \cdots - a_n v_n$ and thus $v_{n+1} = \frac{-a_1}{a_{n+1}} v_1 + \cdots + \frac{-a_n}{a_{n+1}} v_n$ where at least one of $\frac{-a_1}{a_{n+1}}, \ldots, \frac{-a_n}{a_{n+1}}$ is not zero. Thus $v_{n+1} \in \text{Span}\{v_1, \ldots, v_n\}$, a contradiction. Since we reached a contradiction both when $a_{n+1} = 0$ and when $a_{n+1} \neq 0$, our original assumption of $\{v_1, \ldots, v_n, v_{n+1}\}$ being linearly dependent is false, and thus $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly independent. $\square$.

Corollary 10: Let V be a vector space, let $S \subset V$ be a (not necessarily finite) set of

linearly independent vectors, let $v \in V$. Then the set $S \cup \{v\}$ is linearly independent if and only if $v \notin \text{Span}(S)$.

Examples:

1. Let $V = \mathbb{R}^n$, let e_i be the vector with a 1 in the i-th entry and 0 in all other entries, so for $i = 1, \dots, n$ we have $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. Then $e_1, \dots, e_n$ are linearly independent and $\mathbb{R}^n = \text{Span}\{e_1, \dots, e_n\}$. These $e_1, \dots, e_n$ are known as the standard generators of $\mathbb{R}^n$.

2. Let $V = M_2(\mathbb{C})$, $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then M_1 and M_2 are linearly independent, since having scalars $a, b \in \mathbb{C}$ with $aM_1 + bM_2 = \vec{0}$ means:

$$a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ namely } \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

hence $a + b = 0$ and $a - b = 0$, so $a = b = 0$.

3. Let $V = \mathbb{R}[x]$. Then $\{1, x, x^2, \dots\}$ is a linearly independent set. This can be proven by induction: suppose that there are exponents $n \in \mathbb{N}$ with x^n being a linear combination of monomials of lower degree, namely $x^n \in \text{Span}\{1, x, \dots, x^{n-1}\}$.

First, if $n = 1$, this would mean $x \in \text{Span}\{1\}$. Thus $x = a \cdot 1$ for some non-zero coefficient $a \in \mathbb{R}$. Evaluating at $x = 0$ this gives $0 = a$, a contradiction.

Hence $x \notin \text{Span}\{1\}$ and thus by Theorem 9 the set $\{1, x\}$ is linearly independent.

Suppose the statement is true for $n-1$, namely the set $\{1, x, \dots, x^{n-1}\}$ is linearly independent. We now prove the statement for n.

Suppose that $x^n \in \text{Span}\{1, x, \dots, x^{n-1}\}$. Then for some non-zero coefficients $a_0, \dots, a_{n-1} \in \mathbb{R}$ we have $x^n = a_0 \cdot 1 + a_1 \cdot x + \cdots + a_{n-1} \cdot x^{n-1}$. Evaluating at $x = 0$ this gives $0 = a_0$, so in fact we have $x^n = a_1 \cdot x + \cdots + a_{n-1} \cdot x^{n-1}$. This yields $x \cdot x^{n-1} = x \cdot (a_1 + \cdots + a_{n-1} \cdot x^{n-2})$ and thus $x^{n-1} = a_1 + \cdots + a_{n-1} \cdot x^{n-2}$. Hence $x^{n-1} \in \text{Span}\{1, x, \dots, x^{n-2}\}$, so by Theorem 9 the set $\{1, x, \dots, x^{n-1}\}$ is linearly dependent. This is a contradiction with the induction hypothesis, which was that $\{1, x, \dots, x^{n-1}\}$ is linearly independent. Thus our original assumption of $x^n \in \text{Span}\{1, x, \dots, x^{n-1}\}$ is false, meaning that $x^n \notin \text{Span}\{1, x, \dots, x^{n-1}\}$.

Hence by Theorem 9 the set $\{1, x, \dots, x^n\}$ is linearly independent.

To finish the reasoning, suppose that $\{1, x, x^2, \dots\}$ is linearly dependent, we want to achieve a contradiction. Then there are non-zero coefficients $a_{i_1}, \dots, a_{i_n} \in \mathbb{R}$ such that $a_{i_1} \cdot x^{i_1} + \cdots + a_{i_n} \cdot x^{i_n} = 0$, which can be rearranged

$x^{i_n} = -a_{i_1} \cdot x^{i_1} - \cdots - a_{i_{n-1}} \cdot x^{i_{n-1}}$. Hence $x^{i_n} \in \text{Span}\{x^{i_1}, \dots, x^{i_{n-1}}\}$

as $x^{i_n} = \frac{1}{a_{i_n}} \cdot x^{i_1} + \cdots + \frac{a_{i_{n-1}}}{a_{i_n}} \cdot x^{i_{n-1}}$. Hence $x^{i_n} \in \text{Span}\{x^{i_1}, \ldots, x^{i_{n-1}}\}$, in particular $x^{i_n} \in \text{Span}\{1, x, \ldots, x^{i_n - 1}\}$, so by Theorem 9 the set $\{1, x, \ldots, x^{i_n - 1}, x^{i_n}\}$ is linearly dependent, so the set $\{1, x, \ldots, x^{i_n - 1}, x^{i_n - 1 + 1}, \ldots, x^{i_n}\}$ is linearly dependent. This is a contradiction, since we just proved that the set $\{1, x, \ldots, x^n\}$ is linearly independent for all $n \in \mathbb{N}$. Thus our assumption that the set $\{1, x, x^2, \ldots\}$ is linearly dependent is false, meaning that the set $\{1, x, x^2, \ldots\}$ is linearly independent.

4. The vectors $\{(-2,0,3), (1,3,0), (2,4,-1)\}$ are linearly dependent. We consider the matrix with columns $(-2,0,3), (1,3,0), (2,4,-1)$, we row reduce, and we obtain:

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 4 \\ 3 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 3 & 0 & -1 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix in row-reduced echelon form is not the identity matrix, so the vectors $(-2,0,3), (1,3,0), (2,4,-1)$ are not linearly independent by Theorem 8.

5. Solve the system of linear equations:

$$\begin{aligned} 3x_1 - 7x_2 + 4x_3 &= 10 \\ x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 - 2x_3 &= 6. \end{aligned}$$

We can rewrite this system as an equation of matrices:

$$\begin{bmatrix}3\\1\\2\end{bmatrix}x_1+\begin{bmatrix}-7\\-1\\-2\end{bmatrix}x_2+\begin{bmatrix}4\\1\\-2\end{bmatrix}x_3=\begin{bmatrix}10\\3\\6\end{bmatrix} \quad \text{equivalently} \quad \begin{bmatrix}3&-7&4\\1&-2&1\\2&-1&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}10\\3\\6\end{bmatrix}.$$

This system will have a unique solution if and only if the vectors $(3,1,2)$, $(-7,-1,-2)$, $(4,1,2)$ are linearly independent. We use row reduction to find it.

$$\left[\begin{array}{ccc|c}3&-7&4&10\\1&-2&1&3\\2&-1&-2&6\end{array}\right] \xrightarrow[R_1-R_3]{R_3-2R_2} \left[\begin{array}{ccc|c}1&-6&6&4\\1&-2&1&3\\0&3&-4&0\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c}1&-6&6&4\\0&4&-5&-1\\0&3&-4&0\end{array}\right] \xrightarrow[R_2-R_3]{R_1+2R_3}$$

$$\left[\begin{array}{ccc|c}1&0&-2&4\\0&1&-1&-1\\0&3&-4&0\end{array}\right] \xrightarrow[-R_3]{R_3-3R_2} \left[\begin{array}{ccc|c}1&0&-2&4\\0&1&-1&-1\\0&0&1&-3\end{array}\right] \xrightarrow[R_2+R_3]{R_1+2R_3} \left[\begin{array}{ccc|c}1&0&0&-2\\0&1&0&-4\\0&0&1&-3\end{array}\right].$$

Hence $x_1=-2$, $x_2=-4$, $x_3=-3$.

In fact, we always have $V=\text{Span}(V)$ for all vector spaces V. The question to ask is whether there is a subset $S\subset V$ such that $V=\text{Span}(S)$ and such that S has the minimum possible number of elements. We call those subsets a basis of the vector space V.

Definition: Let V be a vector space. A subset $\{v_1,v_2,\dots\}\subset V$ is called a basis of V when $V=\text{Span}\{v_1,v_2,\dots\}$ and $\{v_1,v_2,\dots\}$ is linearly independent.

Examples:

1. Let $V=\mathbb{F}^n$, let e_i be the vector with a 1 in the i-th entry and 0 in all other

entries, so for $i=1,\dots,n$ we have $e_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \dots, e_n=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$. Then $e_1,\dots,e_n$ are linearly independent and $\mathbb{F}^n = \text{Span}\{e_1,\dots,e_n\}$, so $\{e_1,\dots,e_n\}$ is a basis of $\mathbb{F}^n$.

2. Let $V=\mathbb{F}[x]$, the set $\{1,x,x^2,\dots\}$ is a basis of $\mathbb{F}[x]$.

3. Let $V=\mathbb{F}_n[x]$ be the vector space of polynomials of degree at most n, the set $\{1,x,\dots,x^n\}$ is a basis of $\mathbb{F}_n[x]$.

4. Let $V=M_{n\times m}(\mathbb{F})$, let E_{ij} be the matrix with a 1 in the ij-th entry and 0 in all other entries. The set $\{E_{ij}\}_{\substack{1\le j\le m\\ 1\le i\le n}}$ is a basis of $M_{n\times m}(\mathbb{F})$.

<u>Theorem 11:</u> Let V be a vector space. Let β be a finite set that is also a basis of V. Let β' be another basis of V. Then β and β' have the same number of elements, namely $|\beta|=|\beta'|$.

The above result is a consequence of the Replacement Theorem and of Theorem 9. In fact, Theorem 11 also holds when β is infinite. This enables the following definition.

<u>Definition:</u> Let V be a vector space, let β be a basis of V. The <u>dimension</u> of V, denoted $\dim_{\mathbb{F}}(V)$, is the number of elements in β.

A vector space can be finite dimensional or infinite dimensional.

Examples:

1. $\dim_{\mathbb{R}}(\mathbb{R}^n) = n$.

2. $\dim_{\mathbb{R}}(\mathbb{C}^n) = 2n$, $\dim_{\mathbb{C}}(\mathbb{C}^n) = n$.

3. $\dim_{\mathbb{F}}(M_{n\times m}(\mathbb{F})) = n \cdot m$.

4. $\dim_{\mathbb{F}}(\mathbb{F}[x]) = |\mathbb{N}|$.

Theorem 12: Let V be a vector space. The set $\{v_1, \dots, v_n\}$ is a basis of V if and only if every $v \in V$ can be expressed as a linear combination $v = a_1 v_1 + \dots + a_n v_n$ with $a_1, \dots, a_n \in \mathbb{F}$ in an unique way. Namely if $a_1 v_1 + \dots + a_n v_n = b_1 v_1 + \dots + b_n v_n$ for $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{F}$ then $a_1 = b_1, \dots, a_n = b_n$.

We can now answer the question of the existence of basis: every vector space has a basis.

Theorem 13: Let V be a vector space, $V = \operatorname{Span}\{v_1, \dots, v_n\}$. Then there exists a subset $\beta \subseteq \{v_1, \dots, v_n\}$ that is a basis of V.

Proof: We prove this by starting with the empty set, and adding one vector at a time. Let n be the number of vectors spanning V, namely $V = \operatorname{Span}\{v_1, \dots, v_n\}$. If $n = 0$

then $V = \text{Span}\{\}$. Now $V = 0$ and $\beta = \{\}$ is a basis. If $n = 1$ then $V = \text{Span}\{v_1\}$.

Now if $v_1 = \vec{0}$ then $V = 0$ and $\beta = \{\}$ is a basis, and if $v_1 \neq \vec{0}$ then $V \neq 0$ and $\beta = \{v_1\}$ is a basis. If $n \neq 0, 1$ then $V = \text{Span}\{v_1, \dots, v_n\}$. Consider $\text{Span}\{v_1\} \subseteq V$, if $\text{Span}\{v_1\} = V$ then $\beta = \{v_1\}$ is a basis. If $\text{Span}\{v_1\} \neq V$ check whether v_2 is in $\text{Span}\{v_1\}$. If $v_2 \notin \text{Span}\{v_1\}$ then consider $\text{Span}\{v_1, v_2\} \subseteq V$ with $\beta = \{v_1, v_2\}$ as potential basis. If $v_2 \in \text{Span}\{v_1\}$ check whether v_3 is in $\text{Span}\{v_1\}$.

Repeating this process, we obtain at every step a subset $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} \subseteq \{v_1, \dots, v_n\}$ that is linearly independent. Moreover since at every step we always have that the vectors ommited in the candidate basis are in the span of the candidate basis, $\text{Span}\{v_{i_1}, \dots, v_{i_k}\} \supseteq \text{Span}\{v_1, \dots, v_n\} = V$ and thus $V = \text{Span}\{v_{i_1}, \dots, v_{i_k}\}$ so $\beta = \{v_{i_1}, \dots, v_{i_k}\}$ is a basis of V. Since we are only checking a finite number of conditions, because $\{v_1, \dots, v_n\}$ is finite, this process terminates. $\square$.

This proves that a finitely generated vector space has a basis. It does not say that infinitely generated vector spaces have a basis; although that statement is true, it requires a different proof.

Remark: There are several important results related to Theorem 13.

1. Any finite spanning set can be reduced to a basis.

2. Any spanning set with more vectors than $\dim_{\mathbb{F}}(V)$ is not a basis.

3. Any set with fewer vectors than $\dim_{\mathbb{F}}(V)$ does not span V.

Example: Let $V = \mathbb{R}^3$, let $S = \{(2,-3,5),(8,-12,20),(1,0,-2),(0,2,-1),(7,2,0)\}$. Check if we can extract a basis of V from S, and do so.

Since $\mathbb{R}^3$ has dimension 3, a basis for $\mathbb{R}^3$ has exactly 3 vectors. To build a basis, we follow the proof of Theorem 13:

Step 1: Pick two vectors in S that are not linearly dependent.

Since $(2,-3,5) \neq a\cdot(1,0,-2)$ for all $a \in \mathbb{R}$, these are enough.

Step 2: Pick one vector in S that is not in $\text{Span}\{(2,-3,5),(1,0,-2)\}$. We can simplify this a bit further using row reduction:

$$\begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & -2 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 0 & -3 & 9 \\ 1 & 0 & -2 \end{bmatrix} \xrightarrow{\frac{-1}{3}R_1} \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

and thus $\text{Span}\{(2,-3,5),(1,0,-2)\} = \text{Span}\{(0,1,-3),(1,0,-2)\}$ so we immediately see that:

$$(8,-12,20) \in \text{Span}\{(0,1,-3),(1,0,-2)\},$$

$(0,2,-1), (7,2,0) \notin \text{Span}\{(0,1,-3),(1,0,-2)\}$.

Thus $\beta = \{(2,-3,5),(1,0,-2),(0,2,-1)\}$ and $\beta' = \{(2,-3,5),(1,0,-2),(7,2,0)\}$ are basis of $\mathbb{R}^3$.

* Aside on cosets.

A fundamental construction in mathematics is the concept of equivalence, which gives relations between objects. One of the embodyments of equivalence relations are cosets.

Definition: Let V be a vector space, let $W \subseteq V$ be a vector subspace, let $v \in V$. The set

$v + W = \{v + w \mid w \in W\}$ is called a **coset**.

In general cosets are not vector subspaces of V, just subsets.

Definition: Let V be a vector space, let $W \subseteq V$ be a vector subspace. The set formed by the sets $v + W$ for $v \in V$ is called the **quotient space** of V modulo W, denoted V/W. Namely, $V/W = \{v + W \mid v \in V\}$.

Theorem 14: Let V be a vector space, let $W \subseteq V$ be a vector subspace. The set V/W is a vector space over $\mathbb{F}$ with the operations:

$$+ : V/W \times V/W \longrightarrow V/W \qquad (v_1 + W, v_2 + W) \mapsto (v_1 + v_2) + W)$$

and

$$\cdot : \mathbb{F} \times V/W \longrightarrow V/W \qquad (a, v + W) \longmapsto (a \cdot v) + W)$$

Example: Let $V=\mathbb{R}^3$, let $W=\operatorname{Span}\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}=\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\,\middle|\, r\in\mathbb{R}\right\}$. We know that any vector v in $\mathbb{R}^3$ has the form $\begin{bmatrix}a\\b\\c\end{bmatrix}$ with $a,b,c\in\mathbb{R}$, so such a vector is in W if and only if $b=c=0$. Now:

$$v+W=\left\{\begin{bmatrix}a\\b\\c\end{bmatrix}+\begin{bmatrix}r\\0\\0\end{bmatrix}\,\middle|\, r\in\mathbb{R}\right\}=\left\{\begin{bmatrix}a+r\\b\\c\end{bmatrix}\,\middle|\, r\in\mathbb{R}\right\}.$$

In particular if $v\in W$ then it has the form $\begin{bmatrix}a\\0\\0\end{bmatrix}$ for some $a\in\mathbb{R}$, hence:

$$v+W=\left\{\begin{bmatrix}a\\0\\0\end{bmatrix}+\begin{bmatrix}r\\0\\0\end{bmatrix}\,\middle|\, r\in\mathbb{R}\right\}=\left\{\begin{bmatrix}a+r\\0\\0\end{bmatrix}\,\middle|\, r\in\mathbb{R}\right\}=\left\{\begin{bmatrix}s\\0\\0\end{bmatrix}\,\middle|\, s\in\mathbb{R}\right\}=W.$$

The space V/W is formed by the sets:

$$\begin{bmatrix}0\\b\\c\end{bmatrix}+W=\left\{\begin{bmatrix}r\\b\\c\end{bmatrix}\,\middle|\, r\in\mathbb{R}\right\}\qquad \text{for each choice of } b,c\in\mathbb{R}.$$

Now V/W is spanned by the two sets $\begin{bmatrix}0\\1\\0\end{bmatrix}+W$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}+W$, and they are linearly independent since for all $s\in\mathbb{R}$:

$$s\cdot\left(\begin{bmatrix}0\\1\\0\end{bmatrix}+W\right)=\left(s\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}\right)+W=\begin{bmatrix}0\\s\\0\end{bmatrix}+W \quad \text{is different to} \quad \begin{bmatrix}0\\0\\1\end{bmatrix}+W.$$

Hence $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}+W,\ \begin{bmatrix}0\\0\\1\end{bmatrix}+W\right\}$ is a basis of V/W.

Moreover, the set W is the zero vector in V/W.

End of the aside. *

Theorem 15: (Replacement Theorem). Let V be a vector space, $V = \text{Span}\{v_1, \dots, v_n\}$, let $\{u_1, \dots, u_m\} \subset V$ be linearly independent. Then:

1) The cardinality of $\{u_1, \dots, u_m\}$ is, at most, the cardinality of $\{v_1, \dots, v_n\}$.

Namely $m \leq n$.

2) There is a subset $\{v_{i_1}, \dots, v_{i_{n-m}}\} \subseteq \{v_1, \dots, v_n\}$ containing $n-m$ vectors

such that $V = \text{Span}\{u_1, \dots, u_m, v_{i_1}, \dots, v_{i_{n-m}}\}$.

Proof: We use induction on m. Suppose first that $m = 0$, now $0 \leq n$ and setting

$v_{i_j} = v_j$ for $j = 1, \dots, n$ then $V = \text{Span}\{v_1, \dots, v_n\} = \text{Span}\{v_{i_1}, \dots, v_{i_{n-0}}\}$, as

desired. Suppose second that the statement holds for $m-1$, namely given any

linearly independent set with $m-1$ elements $\{w_1, \dots, w_{m-1}\}$ then $m-1 \leq n$ and

there is a subset $\{v_{i_1}, \dots, v_{i_{n-(m-1)}}\} \subset \{v_1, \dots, v_n\}$ such that

$V = \text{Span}\{w_1, \dots, w_{m-1}, v_{i_1}, \dots, v_{i_{n-(m-1)}}\}$. We now prove that the statement

holds for m. Let $\{u_1, \dots, u_m\} \subset V$ be linearly independent, then $\{u_1, \dots, u_{m-1}\}$

is linearly independent, so by induction hypothesis we have $m-1 \leq n$ and

a subset $\{v_{i_1}, \dots, v_{i_{n-m+1}}\} \subset \{v_1, \dots, v_n\}$ with

$V = \text{Span}\{u_1, \dots, u_{m-1}, v_{i_1}, \dots, v_{i_{n-m+1}}\}$. Since $u_m \in V$, we can write:

$$u_m = a_1 u_1 + \cdots + a_{m-1} u_{m-1} + a_m v_{i_1} + \cdots + a_n v_{i_{n-m+1}}$$

where $a_1, \ldots, a_m \in \mathbb{F}$, at least one of them is not zero. If $n = m-1$ then this can be simplified to $u_m = a_1 u_1 + \cdots + a_{m-1} u_{m-1}$, where at least one of the coefficients is not zero. Thus $u_m \in \text{Span}\{u_1, \ldots, u_{m-1}\}$ so by Theorem 9 then $\{u_1, \ldots, u_{m-1}, u_m\}$ is linearly dependent, a contradiction. If $n \neq m-1$, since we know $m-1 \leq n$, then $m-1 < n$ so $m \leq n$, giving the first part of the result. Moreover if all the coefficients $a_m, \ldots, a_n$ are zero then $u_m = a_1 u_1 + \cdots + a_{m-1} u_{m-1}$, where at least one of the coefficients is not zero, giving a contradiction as before. Thus at least one of the coefficients $a_m, \ldots, a_n$ is not zero, say $a_j \neq 0$ for $m \leq j \leq n$. Thus we can rearrange $u_m = a_1 u_1 + \cdots + a_{m-1} u_{m-1} + a_m v_{i_1} + \cdots + a_n v_{i_{n-m+1}}$ as:

$$v_{i_j} = -\frac{a_1}{a_j} u_1 - \cdots - \frac{a_{m-1}}{a_j} u_{m-1} + \frac{1}{a_j} u_m$$
$$- \frac{a_m}{a_j} v_{i_1} - \cdots - \frac{a_{(m-1)+(j-1)}}{a_j} v_{i_{j-1}} - \frac{a_{(m-1)+(j+1)}}{a_j} v_{i_{j+1}} - \cdots - \frac{a_n}{a_j} v_{i_{n-m+1}}.$$

Hence $v_{i_j} \in \text{Span}\{u_1, \ldots, u_m, v_{i_1}, \ldots, v_{i_{j-1}}, v_{i_{j+1}}, \ldots, v_{i_{n-m+1}}\}$, meaning that $\{u_1, \ldots, u_m, v_{i_1}, \ldots, v_{i_{n-m+1}}\} \subset \text{Span}\{u_1, \ldots, u_m, v_{i_1}, \ldots, v_{i_{j-1}}, v_{i_{j+1}}, \ldots, v_{i_{n-m+1}}\}$, so

$V = \text{Span}\{u_1, \ldots, u_m, v_{i_1}, \ldots, v_{i_{n-m+1}}\} \subset \text{Span}\{u_1, \ldots, u_m, v_{i_1}, \ldots, v_{i_{j-1}}, v_{i_{j+1}}, \ldots, v_{i_{n-m+1}}\}$

and of course $\text{Span}\{u_1,\dots,u_m,v_{i_1},\dots,v_{i_{j-1}},v_{i_{j+1}},\dots,v_{i_{n-m+1}}\} \subseteq V$, whence

$V = \text{Span}\{u_1,\dots,u_m,v_{i_1},\dots,v_{i_{j-1}},v_{i_{j+1}},\dots,v_{i_{n-m+1}}\}$. Since the set

$\{v_{i_1},\dots,v_{i_{j-1}},v_{i_{j+1}},\dots,v_{i_{n-m+1}}\}$ has $(n-m+1)-1 = n-m$ elements and is a subset

of $\{v_1,\dots,v_n\}$, this yields the second part of the result. $\square$.

Examples:

1. Let $V = \mathbb{R}^3$, note that $V = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. The set $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ is linearly independent. Now Theorem 15 says that we can find a subset of $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ that complements $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$, that is, two of the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}$ together with $\begin{bmatrix}1\\1\\0\end{bmatrix}$ span $\mathbb{R}^3$. There is more than one choice:

   (a) $\mathbb{R}^3 \neq \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$.

   (b) $\mathbb{R}^3 = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$.

   (c) $\mathbb{R}^3 = \text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$.

2. Let $V = M_{2\times 2}(\mathbb{Z}_3)$ be the vector space of 2 by 2 matrices with entries in $\mathbb{Z}_3$. Now $V = \text{Span}\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$. The set $\left\{\begin{bmatrix}1&2\\0&0\end{bmatrix},\begin{bmatrix}0&0\\2&0\end{bmatrix}\right\}$ is linearly independent, and there are two subsets of the

original generating set that complement it:

(a) $M_{2\times 2}(\mathbb{F}) = \text{Span}\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&2\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\2&0\end{bmatrix}\right\}$.

(b) $M_{2\times 2}(\mathbb{F}) = \text{Span}\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&2\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\2&0\end{bmatrix}\right\}$.

Corollary 16: (Theorem 11) Let V be a vector space. Let β be a finite set that is also a basis of V. Let β' be another basis of V. Then β and β' have the same number of elements, namely $|\beta| = |\beta'|$.

Proof: Since β is a basis then $V = \text{Span}(\beta)$, since β' is a basis then β' is linearly independent, so by Theorem 15 then $|\beta'| \leq |\beta|$. Exchanging the roles of β and β' gives $|\beta| \leq |\beta'|$. Thus $|\beta| = |\beta'|$. □.

Remark: Given a vector space V of dimension $n \in \mathbb{N}$, then:

1. A basis β of V has exactly n elements.

2. A subset of V spanning V has at least n elements.

3. A linearly independent subset of V has at most n elements.

In particular any spanning set of V with exactly n vectors is a basis of V.

Examples:

1. $\dim_{\mathbb{R}}(\mathbb{R}^n) = n$ with basis $\beta = \{e_i = (0,\dots,\overset{i}{1},\dots,0) \mid 1 \leq i \leq n\}$.

2.1. $\dim_{\mathbb{C}}(\mathbb{C}^n) = n$ with basis $\beta = \{ e_i = (0,\dots,\overset{i}{1},\dots,0) \mid 1 \le i \le n \}$.

2.2. $\dim_{\mathbb{R}}(\mathbb{C}^n) = 2n$ with basis

$$\beta = \{ e_i = (0,\dots,\overset{i}{1},\dots,0) \mid 1 \le i \le n \} \cup \{ f_j = (0,\dots,\overset{j}{i},\dots,0) \mid 1 \le j \le n \}.$$

3. $\dim_{\mathbb{F}}(M_{n\times m}(\mathbb{F})) = n \cdot m$ with basis $\beta = \{ E_{ij} \mid 1 \le i \le n,\ 1 \le j \le m \}$.

4. $\dim_{\mathbb{F}}(\mathbb{F}[x]) = |\mathbb{N}|$ with basis $\beta = \{ x^i \mid i \in \mathbb{N} \}$.

5. $\dim_{\mathbb{F}}(\mathbb{F}_n[x]) = n+1$ with basis $\beta = \{ x^i \mid 0 \le i \le n \}$.

A vector subspace W of a vector space V is a vector space on its own, so it will have a basis and a dimension. However, our options will be restricted by what is a basis in V, and by the dimension of V.

Theorem 17: Let V be a finite dimensional vector space, $W \subseteq V$ a vector subspace. Then:

(1) $\dim_{\mathbb{F}}(W) \le \dim_{\mathbb{F}}(V)$,

(2) $\dim_{\mathbb{F}}(W) = \dim_{\mathbb{F}}(V)$ if and only if $V = W$.

In particular, W is finite dimensional.

Proof: (1) Let β be a basis of W. Since β is linearly independent in W it is also linearly independent in V. Now V can be generated with $\dim_{\mathbb{F}}(V)$ vectors, so by Theorem 15 then $\dim_{\mathbb{F}}(W) = |\beta| \le \dim_{\mathbb{F}}(V)$.

(2) We want to prove that $\dim_{\mathbb{F}}(W) = \dim_{\mathbb{F}}(V)$ if and only if $V = W$.

($\Leftarrow$) Suppose $V = W$, then $\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(W)$.

($\Rightarrow$) Suppose $\dim_{\mathbb{F}}(W) = \dim_{\mathbb{F}}(V)$, let β be a basis of W, let γ be a basis of V. By Theorem 15 there is a subset S of γ containing $\dim_{\mathbb{F}}(V) - \dim_{\mathbb{F}}(W)$ vectors such that $V = \text{Span}(\beta \cup S)$. Since $\dim_{\mathbb{F}}(W) = \dim_{\mathbb{F}}(V)$ then S contains zero vectors, so $S = \emptyset$ and $\beta \cup S = \beta$. Hence $V = \text{Span}(\beta) = W$. $\square$.

Thus finite dimensional vector spaces can only have finite dimensional subspaces.

Inspecting the proof of Theorem 17, we find a corollary of the Replacement Theorem.

Corollary 18: Let V be a finite dimensional vector space, $W \subsetneq V$ a vector subspace with basis β. Then we can extend β to a basis of V.

Proof: Let γ be a basis of V. By Theorem 15 there is a subset S of γ containing $\dim_{\mathbb{F}}(V) - \dim_{\mathbb{F}}(W)$ vectors such that $V = \text{Span}(\beta \cup S)$. Note that $S \neq \emptyset$ since $W \neq V$. Now $\beta \cup S$ is a set with at most $\dim_{\mathbb{F}}(V)$ elements that generates V. By the Remark after Corollary 16 then $\beta \cup S$ has exactly $\dim_{\mathbb{F}}(V)$ elements, and thus is a basis of V. $\square$.

Remark: If W is a vector subspace of V with basis $\{w_1, \dots, w_k\}$, we can add vectors $v_{k+1}, \dots, v_n$ to it so that $\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$ is a basis of V.

Now setting $W^c = \text{Span}\{v_{k+1}, \dots, v_n\}$ the complement subspace of W we find $V = W \oplus W^c$.

2. Linear transformations.

We have given the structure of a vector space (over a field) to some sets, we now investigate how different objects with this structure interact. Namely, we are interested in relations between vector spaces, and in how to obtain information about one if we know information about another. We will achieve this by using functions that preserve this vector space structure, the so called linear transformations.

Definition: Let V and W be vector spaces over the same field $\mathbb{F}$. A linear transformation is a function $T: V \to W$ satisfying for all $x, y \in V$ and $a \in \mathbb{F}$:

(1) $T(x+y) = T(x) + T(y)$

(2) $T(a \cdot x) = a \cdot T(x)$.

Remark: In particular, a linear transformation $T: V \to W$ preserves linear combinations, namely $T\left(\sum_{i=1}^{n} a_i \cdot v_i\right) = \sum_{i=1}^{n} a_i \cdot T(v_i)$ for all $v_1, \dots, v_n \in V$ and all

$a_1, \ldots, a_n \in \mathbb{F}$. A linear transformation also sends the zero in V to the zero in W: $T(\vec{0}) = \vec{0}$.

Theorem 19: Let V and W be vector spaces over the same field $\mathbb{F}$.

1) Let $T_1 : V \to W$ and $T_2 : V \to W$ be linear transformations, then:

$$T_1 + T_2 : V \longrightarrow W \qquad \text{is a linear transformation.}$$
$$x \mapsto T_1(x) + T_2(x)$$

2) Let $T : V \to W$ be a linear transformation and $c \in \mathbb{F}$, then:

$$c \cdot T : V \longrightarrow W \qquad \text{is a linear transformation.}$$
$$x \longmapsto c \cdot T(x)$$

3) The set $\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is a linear transformation}\}$ is a vector space over $\mathbb{F}$ with the operations above.

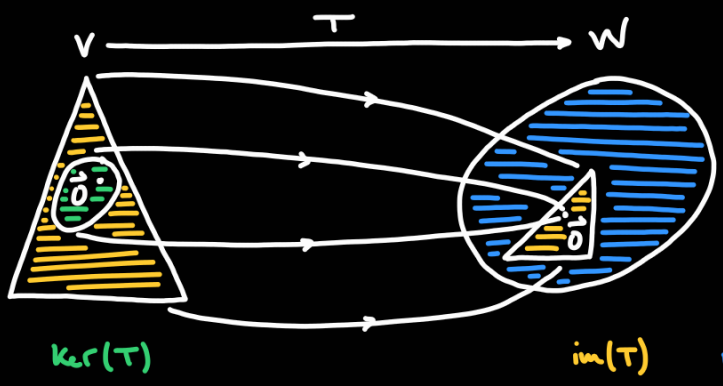
Proof: Straightforward, use the definition for 1) and 2), use Theorem 4 for 3). □.

Definition: Let V and W be vector spaces over the same field $\mathbb{F}$, let $T : V \to W$ be a linear transformation. The kernel or null space of T, denoted $\ker(T)$, is the set:

$$\ker(T) = \{x \in V \mid T(x) = \vec{0}\}.$$

The image or range of T, denoted $\text{im}(T)$, is the set:

$$\text{im}(T) = \{y \in W \mid \text{there exists } x \in V \text{ with } T(x) = y\} = \{T(x) \in W \mid x \in V\}.$$



Examples:

1. A function T between a set S and a field $\mathbb{F}$, namely $T \in \mathcal{F}(S, \mathbb{F})$, that is a polynomial of degree less than or equal to 1 is a linear transformation on every vector space structure on S.

2. The function $T: M_{n\times m}(\mathbb{F}) \longrightarrow M_{n\times m}(\mathbb{F})$ is a linear transformation since
$$A \longmapsto A^t$$
$T(A+B) = (A+B)^t = A^t + B^t = T(A) + T(B)$ and $T(c\cdot A) = (c\cdot A)^t = c\cdot A^t = c\cdot T(A)$.

3. The function $T: \mathbb{F}_n[x] \longrightarrow \mathbb{F}_{n-1}[x]$ is a linear transformation since
$$f \longmapsto \frac{d}{dx} f$$
$T(f+g) = \frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g = T(f) + T(g)$ and

$T(c\cdot f) = \frac{d}{dx}(c\cdot f) = c\cdot \frac{d}{dx} f = c\cdot T(f)$.

4. The function $T: \mathcal{C}(\mathbb{R}, \mathbb{R}) \longrightarrow \mathbb{R}$ is a linear transformation.
$$f \longmapsto \int_a^b f\, dx$$

5. The functions from $\mathbb{R}$ to $\mathbb{R}$ given by $T(x) = x^2$, $T(x) = \frac{1}{x}$, $T(x) = e^x$, $T(x) = \sin(x)$, $T(x) = \cos(x)$, and combinations of these, are not linear transformations.

Example: Compute the kernel and image of the linear transformation $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$.

$(x,y,z) \mapsto (x-y, 2z)$

For the kernel, we want $\ker(T) = \{(x,y,z) \mid T(x,y,z) = (0,0)\}$ so:

$(x,y,z) \in \ker(T)$ if and only if $T(x,y,z) = (0,0)$, namely $(x-y, 2z) = (0,0)$

so $x-y = 0$ and $2z = 0$ so $x = y$ and $z = 0$.

Thus (x,y,z) should look like $(x,x,0)$, and $\ker(T) = \{(x,x,0) \mid x \in \mathbb{R}\}$.

For the image, we want $\text{im}(T) = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\}$ so:

$(u,v) \in \text{im}(T)$ if and only if $(u,v) = T(x,y,z) = (x-y, 2z)$,

so $u = x-y$ and $v = 2z$.

Now since any real number u can be obtained as a difference of two real numbers $x-y$, and any real number v can be obtained by doubling another real number z, then u and v take all the possible real values. Hence $\text{im}(T) = \mathbb{R}^2$.

Theorem 20: Let V and W be vector spaces over the same field $\mathbb{F}$, let $T: V \longrightarrow W$ be a linear transformation. Then:

1) $\ker(T)$ is a subspace of V.

2) $\text{im}(T)$ is a subspace of W.

Proof: We use Theorem 4:

1) Since $T(\vec{0}) = \vec{0}$ then $\vec{0} \in \ker(T)$.

Suppose that $x, y \in \ker(T)$, namely $T(x) = \vec{0} = T(y)$. Then:

$T(x+y) = T(x) + T(y) = \vec{0} + \vec{0} = \vec{0}$ so $x+y \in \ker(T)$.

Suppose that $x \in \ker(T)$, so $T(x) = \vec{0}$, and $c \in \mathbb{F}$. Then:

$T(c \cdot x) = c \cdot T(x) = c \cdot \vec{0} = \vec{0}$ so $c \cdot x \in \ker(T)$.

2) Since $T(\vec{0}) = \vec{0}$ then $\vec{0} \in \text{im}(T)$.

Suppose that $u, v \in \text{im}(T)$, so there are $x, y \in V$ with $T(x) = u$ and $T(y) = v$.

Then $T(x+y) = T(x) + T(y) = u + v$ so $u+v \in \text{im}(T)$.

Suppose that $u \in \text{im}(T)$, so there is $x \in V$ with $T(x) = u$, and $c \in \mathbb{F}$. Then:

$T(c \cdot x) = c \cdot T(x) = c \cdot u$ so $c \cdot u \in \text{im}(T)$. $\square$.

Theorem 21: Let V and W be vector spaces over the same field $\mathbb{F}$, let $T: V \longrightarrow W$ be a linear transformation, let $\beta = \{v_1, \dots, v_n\}$ be a basis of V. Then

$T(\beta) = \{T(v_1), \dots, T(v_n)\}$ spans $\text{im}(T)$, namely $\text{im}(T) = \text{Span}\{T(v_1), \dots, T(v_n)\}$.

Proof: We prove the double inclusion of $\text{Span}\{T(v_1), \dots, T(v_n)\}$ in $\text{im}(T)$ and of $\text{im}(T)$ in $\text{Span}\{T(v_1), \dots, T(v_n)\}$.

2) Since $T(v_1), \dots, T(v_n) \in \text{im}(T)$ then $\text{Span}\{T(v_1), \dots, T(v_n)\} \subseteq \text{im}(T)$.

$\subseteq$) Let $u \in \text{im}(T)$, then there exists $v \in V$ with $T(v) = u$. Since β is a basis of V, we can write $v = \sum_{i=1}^{n} a_i \cdot v_i$ for some coefficients $a_1, \dots, a_n \in \mathbb{F}$, and thus:

$$u = T(v) = T\left(\sum_{i=1}^{n} a_i \cdot v_i\right) = \sum_{i=1}^{n} T(a_i \cdot v_i) = \sum_{i=1}^{n} a_i \cdot T(v_i).$$

Hence $u \in \text{Span}\{T(v_1), \dots, T(v_n)\}$. Since this holds for all $u \in \text{im}(T)$, we have that $\text{im}(T) \subseteq \text{Span}\{T(v_1), \dots, T(v_n)\}$. □.

<u>Theorem 22:</u> (Rank-Nullity Theorem) Let V and W be vector spaces over the same field $\mathbb{F}$, with V finite dimensional, and let $T: V \longrightarrow W$ be a linear transformation. Then:

$$\dim(V) = \dim(\ker(T)) + \dim(\text{im}(T)).$$

<u>Proof:</u> Since V is finite dimensional, set $n = \dim(V)$. Writing $k = \dim(\ker(T))$, since $\ker(T)$ is a subspace of V, then $k \leq n$. Given a basis $\{v_1, \dots, v_k\}$ of $\ker(T)$, by Corollary 18 we can extend it to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V. As we remarked this allows us to write $V = \ker(T) \oplus \ker(T)^c$ where $\ker(T)^c$ has basis $\{v_{k+1}, \dots, v_n\}$.

We now claim that $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis of $\text{im}(T)$. To prove this, we have to prove that $\{T(v_{k+1}), \dots, T(v_n)\}$ spans $\text{im}(T)$ and is linearly independent.

To prove spanning, note that since $\{v_1, \dots, v_n\}$ is a basis of V then

$\{T(v_1), \ldots, T(v_n)\}$ spans $\text{im}(T)$ by Theorem 21. Now $v_1, \ldots, v_k \in \ker(T)$ so $T(v_1) = \cdots = T(v_k) = \vec{0}$, which do not contribute to the span. Thus $\{T(v_1), \ldots, T(v_n)\} \in \text{Span}\{T(v_{k+1}), \ldots, T(v_n)\}$ so $\text{im}(T) = \text{Span}\{T(v_1), \ldots, T(v_n)\} \subseteq \text{Span}\{T(v_{k+1}), \ldots, T(v_n)\}$. Since $\text{Span}\{T(v_{k+1}), \ldots, T(v_n)\} \subseteq \text{im}(T)$ then $\text{im}(T) = \text{Span}\{T(v_{k+1}), \ldots, T(v_n)\}$.

To prove linear independence, we proceed by contradiction. Suppose that the set is linear dependent, namely there are scalars $a_{k+1}, \ldots, a_n \in \mathbb{F}$, at least one non-zero, such that $a_{k+1} \cdot T(v_{k+1}) + \cdots + a_n T(v_n) = \vec{0}$. Then $T(a_{k+1} \cdot v_{k+1} + \cdots + a_n v_n) = \vec{0}$ so $a_{k+1} \cdot v_{k+1} + \cdots + a_n v_n \in \ker(T)$. Since $\{v_1, \ldots, v_k\}$ is a basis of $\ker(T)$ then $a_{k+1} \cdot v_{k+1} + \cdots + a_n v_n \in \text{Span}\{v_1, \ldots, v_k\}$, so there are scalars $a_1, \ldots, a_k \in \mathbb{F}$ such that $a_{k+1} \cdot v_{k+1} + \cdots + a_n v_n = a_1 \cdot v_1 + \cdots + a_k \cdot v_k$, which we can rewrite as $-a_1 \cdot v_1 - \cdots - a_k \cdot v_k + a_{k+1} + \cdots + a_n v_n = \vec{0}$. This is a linear combination of the basis $\{v_1, \ldots, v_n\}$ of V that is zero, but at least one of the $a_{k+1}, \ldots, a_n \in \mathbb{F}$ is not zero, contradicting the linear independency of the basis. Thus our assumption that $\{T(v_{k+1}), \ldots, T(v_n)\}$ is linear dependent is false, hence it is linearly

independent.

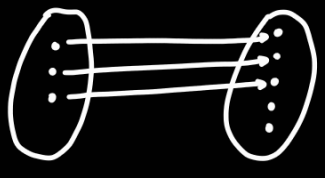
Hence $\{T(\vec{v}_{k+1}), \ldots, T(\vec{v}_n)\}$ is a basis of $\text{im}(T)$, meaning that $\dim(\text{im}(T)) = n-k$.

Now $\dim(\ker(T)) + \dim(\text{im}(T)) = k + (n-k) = n = \dim(V)$. □.

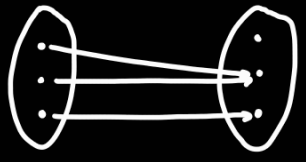
Remark: We call $\dim(\ker(T))$ the nullity of T and $\dim(\text{im}(T))$ the rank of T.

Definition: Let V and W be vector spaces over the same field $\mathbb{F}$, let $T: V \to W$ be a linear transformation. We say that T is injective or one-to-one if $T(x) = T(y)$ implies $x = y$. We say that T is surjective or onto if for every $y \in W$ there is $x \in V$ with $T(x) = y$.

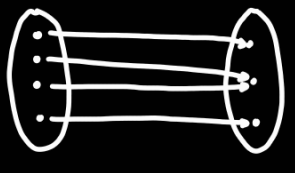
We often write $T: V \hookrightarrow W$ when T is injective, and $T: V \twoheadrightarrow W$ when T is surjective. Pictorially, we have:



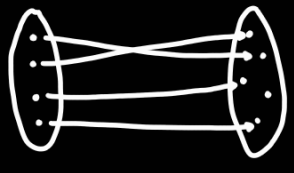
injective, but



not injective, and



surjective, but



not surjective.

Theorem 23: Let V and W be vector spaces over the same field $\mathbb{F}$, let $T: V \to W$ be a linear transformation.

(1) T is injective if and only if $\ker(T) = \{\vec{0}\}$.

(2) T is surjective if and only if $\text{im}(T) = W$.

Proof: (1) ($\Rightarrow$) Suppose T is injective. We prove $\ker(T) = \{\vec{0}\}$. Let $x \in \ker(T)$, so $T(x) = \vec{0}$.

Now $T(x) = \vec{0} = T(\vec{0})$ so by injectivity of T we have $x = \vec{0}$, the sole element in $\ker(T)$.

($\Leftarrow$) Suppose $\ker(T) = \{\vec{0}\}$. We prove T injective. Let $x, y \in V$ such that $T(x) = T(y)$, then $\vec{0} = T(x) - T(y) = T(x-y)$ so $x-y \in \ker(T)$. Hence $x-y = \vec{0}$ so $x = y$, and T is injective.

(2) ($\Rightarrow$) Suppose T is surjective. We prove $\text{im}(T) = W$. Since $\text{im}(T) \subseteq W$, it suffices to prove $W \subseteq \text{im}(T)$. Let $y \in W$, since T is surjective there is $x \in V$ with $T(x) = y$, and thus $y \in \text{im}(T)$.

($\Leftarrow$) Suppose $W = \text{im}(T)$. We prove T surjective. Let $y \in W$, since $W = \text{im}(T)$ then $y \in \text{im}(T)$, so there is $x \in V$ with $T(x) = y$. Thus T is surjective. $\square$.

The concepts of injectivity and surjectivity are different in general, but sometimes they coincide.

Theorem 24: Let V and W be vector spaces over the same field $\mathbb{F}$, let $T: V \longrightarrow W$ be a linear transformation, let $\dim(V) = \dim(W)$ be a finite natural number. The

following are equivalent:

(1) T is injective.

(2) T is surjective.

(3) The dimension of $\text{im}(T)$ equals the dimension of V.

Proof: We have to prove that (1) holds if and only if (2) holds if and only if (3) holds. It is enough to prove the equivalence between (1) and (3), and the equivalence between (2) and (3).

(1) $\Rightarrow$ (3) Suppose T is injective. Using Theorem 23 we know that $\ker(T) = \{\vec{0}\}$. Since $\dim(\ker(T)) = \dim(\{\vec{0}\}) = 0$ by Theorem 22 then $\dim(V) = \dim(\text{im}(T))$.

(3) $\Rightarrow$ (1) Suppose $\dim(V) = \dim(\text{im}(T))$. Since by Theorem 22 then $\dim(V) = \dim(\text{im}(T)) + \dim(\ker(T))$, we have $\dim(\ker(T)) = 0$ and thus $\ker(T) = \text{Span}\{\} = \{\vec{0}\}$. Now by Theorem 23 we have T injective.

(2) $\Rightarrow$ (3) Suppose T is surjective. Using Theorem 23 we know that $\text{im}(T) = W$, so $\dim(V) = \dim(W) = \dim(\text{im}(T))$.

(3) $\Rightarrow$ (2) Suppose $\dim(V) = \dim(\text{im}(T))$ of T [illegible] of

(3) $\Rightarrow$ (2) Suppose $\dim(V) = \dim(\text{im}(T))$. Now $\text{im}(T)$ is a vector subspace of W by Theorem 20, and since $\dim(W) = \dim(V) = \dim(\text{im}(T))$, by Theorem 17 then $W = \text{im}(T)$. Now by Theorem 23 then T is surjective. $\square$.

Given a linear transformation $T: V \to W$, often it is more convenient to compute the dimensions of $\ker(T)$ and $\text{im}(T)$ than to compute the dimensions of V and W. Using Theorem 22 and Theorem 24, we can often determine what we want from just knowing $\ker(T)$ and $\text{im}(T)$.

We will now see that a linear map $T: V \to W$ is completely determined by where it sends a basis of V. This means that to check a characteristic of a linear transformation, such as checking whether T is linear or compute its kernel and image, it suffices to work with a basis of V.

Theorem 25: Let V and W be vector spaces over the same field $\mathbb{F}$, let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V, let $\gamma = \{w_1, \ldots, w_n\}$ be a basis of W. Then the function defined by $T: V \longrightarrow W$ is a linear transformation, and it is unique.
$$\sum_{i=1}^{n} a_i \cdot v_i \mapsto \sum_{i=1}^{n} a_i \cdot w_i$$

Proof: We first prove that $T: V \to W$ is linear. Let $x, y \in V$, since β is a basis of V we can write $x = \sum_{i=1}^{n} a_i \cdot v_i$ and $y = \sum_{i=1}^{n} b_i \cdot v_i$ for some scalars

$a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{F}$. Now:

$$T(x+y) = T\left(\sum_{i=1}^{n} a_i \cdot v_i + \sum_{i=1}^{n} b_i \cdot v_i\right) = T\left(\sum_{i=1}^{n} (a_i + b_i) \cdot v_i\right) = \sum_{i=1}^{n} (a_i + b_i) \cdot w_i =$$

$$= \sum_{i=1}^{n} a_i \cdot w_i + \sum_{i=1}^{n} b_i \cdot w_i = T(x) + T(y).$$

Similarly if $a \in \mathbb{F}$ then:

$$T(a \cdot x) = T\left(a \cdot \sum_{i=1}^{n} a_i \cdot v_i\right) = T\left(\sum_{i=1}^{n} (a \cdot a_i) \cdot v_i\right) = \sum_{i=1}^{n} (a \cdot a_i) \cdot w_i =$$

$$= a \cdot \sum_{i=1}^{n} a_i \cdot w_i = a \cdot T(x).$$

Hence $T: V \to W$ is linear. Suppose that there is another linear transformation $T^1: V \to W$ such that $T^1(v_i) = w_i$ for all $i = 1, \dots, n$. Since any $v \in V$ can be written as $v = \sum_{i=1}^{n} a_i \cdot v_i$ for some $a_1, \dots, a_n \in \mathbb{F}$, then:

$$T(v) = T\left(\sum_{i=1}^{n} a_i \cdot v_i\right) = \sum_{i=1}^{n} a_i \cdot T(v_i) = \sum_{i=1}^{n} a_i \cdot T^1(v_i) = T^1\left(\sum_{i=1}^{n} a_i \cdot v_i\right) = T^1(v).$$

Hence $T(v) = T^1(v)$ for all $v \in V$, so $T = T^1$ and T is unique. □.

Note that we only used that $T: V \to W$ is defined on a basis of V, the values $T(v_1), \dots, T(v_n)$ do not matter. Namely, a linear transformation $T: V \to W$ is uniquely determined by its image on a basis of V.

Example: Consider the function $T: \mathbb{R}_2[x] \longrightarrow \mathbb{R}_3[x]$. Is it linear? Is it

$$f(x) \longmapsto 2 \cdot f'(x) + \int_0^x 3 \cdot f(t)\, dt$$

injective? Is it surjective?

Note that differentiation and integration are linear transformations, and T is obtained by multiplying linear transformations by scalars, and then adding up the remaining linear transformations. Hence T is indeed a linear transformation.

We now compute the dimensions of the image and kernel of T, using that $\{1, x, x^2\}$ is a basis of $\mathbb{R}_2[x]$. We obtain:

$$T(1) = 3x, \quad T(x) = 2 + \frac{3}{2}x^2, \quad T(x^2) = 4x + x^3,$$

so $\text{im}(T) = \text{Span}\{3x, 2+\frac{3}{2}x^2, 4x+x^3\}$. Since $3x, 2+\frac{3}{2}x^2, 4x+x^3$ all have different degrees, they are linearly independent and thus $\dim(\text{im}(T)) = 3$.

Since $\mathbb{R}_3[x]$ has basis $\{1, x, x^2, x^3\}$, then $\dim(\mathbb{R}_3[x]) = 4$ and $\text{im}(T) \subsetneq \mathbb{R}_3[x]$, so T is not surjective. Moreover:

$$3 = \dim(\mathbb{R}_2[x]) = \dim(\text{im}(T)) + \dim(\ker(T)) = 3 + \dim(\ker(T))$$

and thus $\dim(\ker(T)) = 0$ so $\ker(T) = \{\vec{0}\}$ so T is injective.

We are now in a position to justify why "linear transformations are equivalent to matrices". First, we will introduce the notions of coordinate vector. Second, we will use this to define the matrix associated to a linear transformation. Third, we will explain how these constructions enable us to understand a finite dimensional vector

space V, say of dimension n, in terms of $\mathbb{F}^n$. In particular, a linear transformation from V to W will be identified with a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$, where m is the dimension of W, which in turn corresponds to an $m \times n$ matrix.

Definition: Let V be a finite dimensional vector space with basis $\beta = \{v_1, \dots, v_n\}$, let $v \in V$ be written as $v = \sum_{i=1}^{n} a_i \cdot v_i$ for some $a_1, \dots, a_n \in \mathbb{F}$. We say that the coordinate vector of v with respect to the basis β is:

$$[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Example: Let $V = \mathbb{R}_2[x]$, it has basis $\beta = \{1, x, x^2\}$ and $\gamma = \{1+x, 1-x, 3x^2\}$.

Let $f(x) = 3 - 2x + 4x^2$. We can also write $f(x) = \frac{1}{2}(1+x) + \frac{5}{2}(1-x) + \frac{4}{3} \cdot 3x^2$. Hence:

$$[f(x)]_\beta = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \quad \text{and} \quad [f(x)]_\gamma = \begin{bmatrix} 1/2 \\ 5/2 \\ 4/3 \end{bmatrix}.$$

Definition: Let V and W be vector spaces over the same field $\mathbb{F}$, let $\beta = \{v_1, \dots, v_n\}$ be a basis of V, let $\gamma = \{w_1, \dots, w_m\}$ be a basis of W, let $T: V \to W$ be a linear transformation. Write $T(v_j) = \sum_{i=1}^{m} a_{ij} \cdot w_i$ with $a_{ij} \in \mathbb{F}$ for each $i = 1, \dots, m$ and $j = 1, \dots, n$. The matrix associated to T with respect to the basis β and γ is:

$$[T]_\beta^\gamma = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \Big[[T(v_1)]_\gamma \cdots [T(v_n)]_\gamma \Big].$$

We should think of $[v]_\beta$ and $[T]_\beta^\gamma$ as just notation for the linear combinations $v = \sum_{i=1}^{n} a_i \cdot v_i$ and $T(v_j) = \sum_{i=1}^{m} a_{ij} \cdot w_i$ for $j = 1, \ldots, n$. These vectors and matrices encode all the information about v and T. Since we understand matrices well, it will be useful to have access to a matrix containing all the information required to work with abstract v and T.

Theorem 26: Let V and W be vector spaces over the same field $\mathbb{F}$, let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V, let $\gamma = \{w_1, \ldots, w_m\}$ be a basis of W. Let $T: V \longrightarrow W$ and $T': V \longrightarrow W$ be linear transformations, let $c \in \mathbb{F}$. Then:

1) $[T+T']_\beta^\gamma = [T]_\beta^\gamma + [T']_\beta^\gamma$ and

2) $[c \cdot T]_\beta^\gamma = c \cdot [T]_\beta^\gamma$.

Proof: We first establish the notation we will use. Suppose:

$$T(v_j) = \sum_{i=1}^{m} a_{ij} \cdot w_i, \quad T'(v_j) = \sum_{i=1}^{m} b_{ij} \cdot w_i, \quad (T+T')(v_j) = \sum_{i=1}^{m} c_{ij} \cdot w_i,$$

and $(c \cdot T)(v_j) = \sum_{i=1}^{m} d_{ij} \cdot w_i$, now:

$$(T+T')(v_j) = T(v_j) + T'(v_j) = \sum_{i=1}^{m} a_{ij} \cdot w_i + \sum_{i=1}^{m} b_{ij} \cdot w_i = \sum_{i=1}^{m} (a_{ij} + b_{ij}) \cdot w_i$$

$$(c\cdot T)(v_j) = c\cdot T(v_j) = c\cdot \sum_{i=1}^{m} a_{ij}\cdot w_i = \sum_{i=1}^{m} (c\cdot a_{ij})\cdot w_i$$

and thus $c_{ij} = a_{ij} + b_{ij}$ and $d_{ij} = c\cdot a_{ij}$ by Theorem 12. Then:

1) $([T+T']_\beta^\gamma)_{ij} = c_{ij} = a_{ij} + b_{ij} = ([T]_\beta^\gamma)_{ij} + ([T']_\beta^\gamma)_{ij}$ for all $i=1,\dots,m$

and $j=1,\dots,n$. Thus $[T+T']_\beta^\gamma = [T]_\beta^\gamma + [T']_\beta^\gamma$.

2) $([c\cdot T]_\beta^\gamma)_{ij} = d_{ij} = c\cdot a_{ij} = c\cdot([T]_\beta^\gamma)_{ij}$ for all $i=1,\dots,m$ and $j=1,\dots,n$.

Thus $[c\cdot T]_\beta^\gamma = c\cdot[T]_\beta^\gamma$. □.

Theorem 27: Let V, W, X be vector spaces over the same field F, let $\alpha = \{v_1,\dots,v_n\}$, $\beta = \{w_1,\dots,w_m\}$, $\gamma = \{x_1,\dots,x_p\}$ be basis of V, W, X. Let $T: V \to W$ and $T': W \to X$ be linear transformations. Then $T'\circ T: V \to X$ is a linear transformation and $[T'\circ T]_\alpha^\gamma = [T']_\beta^\gamma \cdot [T]_\alpha^\beta$.

Proof: Let $c \in F$, $x, y \in V$, then:

$$(T'\circ T)(x+y) = T'(T(x+y)) = T'(T(x)+T(y)) = T'(T(x)) + T'(T(x)) =$$

$$(T'\circ T)(x) + (T'\circ T)(y),$$

$$(T'\circ T)(c\cdot x) = T'(T(c\cdot x)) = T'(c\cdot T(x)) = c\cdot T'(T(x)) = c\cdot(T'\circ T)(x).$$

Hence $T'\circ T$ is indeed linear. We now establish the notation we will use. Suppose:

$T(v_j) = \sum_{k=1}^{m} b_{kj}\cdot w_k$ and $T'(w_k) = \sum_{i=1}^{p} a_{ik}\cdot x_i$, now:

$$(T' \circ T)(v_j) = T'\left(T(v_j)\right) = T'\left(\sum_{k=1}^{m} b_{kj} \cdot w_k\right) = \sum_{k=1}^{m} b_{kj} \cdot T'(w_k) =$$

$$= \sum_{k=1}^{m} b_{kj} \cdot \left(\sum_{i=1}^{p} a_{ik} \cdot x_i\right) = \sum_{i=1}^{p} \left(\sum_{k=1}^{m} a_{ik} \cdot b_{kj}\right) \cdot x_i .$$

Thus $\left([T' \circ T]_\alpha^\gamma\right)_{ij} = \sum_{k=1}^{m} a_{ik} \cdot b_{kj} = \sum_{k=1}^{m} \left([T']_\beta^\gamma\right)_{ik} \cdot \left([T]_\alpha^\beta\right)_{kj} = \left([T']_\beta^\gamma \cdot [T]_\alpha^\beta\right)_{ij}$

so $[T' \circ T]_\alpha^\gamma = [T']_\beta^\gamma \cdot [T]_\alpha^\beta$. □.

Theorem 28: Let V and W be vector spaces over the same field $\mathbb{F}$, let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V, let $\gamma = \{w_1, \ldots, w_m\}$ be a basis of W. Let $T: V \longrightarrow W$ be a linear transformation. Then $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$ for all $v \in V$.

Proof: We first establish the notation we will use. Suppose:

$$T(v_j) = \sum_{i=1}^{m} b_{ij} \cdot w_i \quad \text{and} \quad v = \sum_{j=1}^{n} a_j \cdot v_j , \quad \text{now:}$$

$$T(v) = T\left(\sum_{j=1}^{n} a_j \cdot v_j\right) = \sum_{j=1}^{n} a_j \cdot T(v_j) = \sum_{j=1}^{n} a_j \cdot \left(\sum_{i=1}^{m} b_{ij} \cdot w_i\right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} b_{ij} \cdot a_j\right) \cdot w_i .$$

Hence $\left([T(v)]_\gamma\right)_i = \sum_{j=1}^{n} b_{ij} \cdot a_j = \sum_{j=1}^{n} \left([T]_\beta^\gamma\right)_{ij} \cdot \left([v]_\beta\right)_j = \left([T]_\beta^\gamma \cdot [v]_\beta\right)_i$, and thus

$[T(v)]_\gamma = [T]_\beta^\gamma \cdot [v]_\beta$. □.

Example: Let $T: \mathbb{R}_2[x] \longrightarrow \mathbb{R}_3[x]$, $f(x) \longmapsto \int_0^x f(t)\,dt$, consider $\beta = \{1, x, x^2\}$ a basis of $\mathbb{R}_2[x]$ and $\gamma = \{1, x, x^2, x^3\}$ a basis of $\mathbb{R}_3[x]$. Now:

$$T(1) = x , \quad T(x) = \frac{x^2}{2} , \quad T(x^2) = \frac{x^3}{3} .$$

Hence:

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}. \quad \text{Not that } [1]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(1)]_\gamma = [x]_\gamma = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and}$$

$$[T]_\beta^\gamma \cdot [1]_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = [T(1)]_\gamma.$$

Consider $\tilde{T}: \mathbb{R}_3[x] \longrightarrow \mathbb{R}_2[x]$, we have:

$$f(x) \longmapsto f'(x)$$

$$\tilde{T}(1) = 0, \quad \tilde{T}(x) = 1, \quad \tilde{T}(x^2) = 2x, \quad \tilde{T}(x^3) = 3x^2.$$

Hence:

$$[\tilde{T}]_\gamma^\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Moreover:

$$\tilde{T} \circ T(1) = 1, \quad \tilde{T} \circ T(x) = x, \quad \tilde{T} \circ T(x^2) = x^2, \text{ so } [\tilde{T} \circ T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now:

$$[\tilde{T}]_\gamma^\beta [T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [T' \circ T]_\beta^\beta.$$

Our approach to the statement "linear transformations are equivalent to matrices" requires us to make this statement formal for $\mathbb{F}^n$ and $\mathbb{F}^m$. The following proves that the function $M_{m \times n}(\mathbb{F}) \longrightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$, $A \longmapsto T_A$ is a linear transformation, injective, and surjective.

Definition: Let $A \in M_{m \times n}(\mathbb{F})$, we say that the linear transformation $T_A: \mathbb{F}^n \longrightarrow \mathbb{F}^m$, $x \longmapsto Ax$

is the left-multiplication by A.

Theorem 29: Let $A \in M_{m\times n}(\mathbb{F})$, then $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ is a linear transformation and:

1) $[T_A]_{\sigma_n}^{\sigma_m} = A$, where σ_n and σ_m are the standard basis of $\mathbb{F}^n$ and $\mathbb{F}^m$.

2) Let $B \in M_{m\times n}(\mathbb{F})$, then $T_A = T_B$ if and only if $A = B$.

3) Let $B \in M_{m\times n}(\mathbb{F})$, then $T_{A+B} = T_A + T_B$.

4) Let $c \in \mathbb{F}$, then $T_{c\cdot A} = c \cdot T_A$.

5) Let $B \in M_{n\times p}(\mathbb{F})$, then $T_{A\cdot B} = T_A \circ T_B$.

6) Let $I_n \in M_{n\times n}(\mathbb{F})$ be the identity matrix, then $T_{I_n} = id_{\mathbb{F}^n}$, where $id_{\mathbb{F}^n}$ is the identity function on $\mathbb{F}^n$ (namely $id_{\mathbb{F}^n}(x) = x$ for all $x \in \mathbb{F}^n$).

Proof: It follows from the definitions of matrix operations, but only basic facts are used, and a straightforward computation suffices. □.

Remark: Let V be a vector space, the identity function on V is the linear transformation that sends every element in V to itself: $id_V : V \longrightarrow V$, $v \longmapsto v$.

To further formalize the notion of "equivalence", we will use invertible functions.

Definition: Let V and W be vector spaces, let $T : V \to W$ be a linear transformation.

We say that a linear transformation $S : W \longrightarrow V$ is the inverse of T when

$ST = id_V$ and $TS = id_W$. We denote the inverse of T by $T^{-1}: W \to V$.

When T has an inverse, we say that T is <u>invertible</u>.

<u>Theorem 30:</u> Let V, W, X be vector spaces over the same field $\mathbb{F}$, let $T: V \to W$ and $U: W \to X$ be invertible linear transformations. Then:

1) The inverse of T is unique.

2) The inverse of UT is $T^{-1}U^{-1}$, namely $(UT)^{-1} = T^{-1}U^{-1}$

3) The inverse of the inverse of T is T, namely $(T^{-1})^{-1} = T$.

<u>Proof:</u> Straightforward using the definitions. □.

<u>Theorem 31:</u> Let V and W be vector spaces over the same field $\mathbb{F}$, let $T: V \to W$ be a linear transformation. Then T is invertible if and only if T is injective and surjective.

<u>Proof:</u> ($\Rightarrow$) Suppose that T is invertible. Then there is a linear transformation $T^{-1}: W \to V$ with $T^{-1}T = id_V$ and $TT^{-1} = id_W$. Suppose that $x, y \in V$ with $T(x) = T(y)$, then:

$$x = id_V(x) = T^{-1}T(x) = T^{-1}(T(x)) = T^{-1}(T(y)) = T^{-1}T(y) = id_V(y) = y$$

so T is injective. Suppose that $z \in W$, then $T^{-1}(z) \in V$ and:

$$T(T^{-1}(z)) = T \circ T^{-1}(z) = id_W(z) = z$$

so setting $x = T^{-1}(z)$ we found an element $x \in V$ such that $T(x) = z$, so T is surjective.

($\Leftarrow$) Suppose that T is injective and surjective. Consider the function of sets $S : W \longrightarrow V$ defined by $S(y) = x$ if and only if $T(x) = y$. This is
$y \longmapsto x$
indeed a function of sets: if $y \in W$ then since T is surjective there is an element $x \in V$ such that $T(x) = y$, and such element x is unique because if there is another $x' \in V$ such that $T(x') = y$ then since T is injective and $T(x) = y = T(x')$ we have $x = x'$. Thus S is well defined, and it is our candidate for the inverse of T: by definition $S \circ T = id_V$ and $T \circ S = id_W$. It only remains to prove that S is a linear transformation.

Let $x, y \in W$ and $c \in \mathbb{F}$, then:

$$T\big(S(x+y)\big) = T \circ S(x+y) = id_W(x+y) = x+y = id_W(x) + id_W(y) =$$

$$T \circ S(x) + T \circ S(y) = T(S(x)) + T(S(y)) = T\big(S(x) + S(y)\big),$$

$$T\big(S(c \cdot x)\big) = T \circ S(c \cdot x) = c \cdot x = c \cdot id_W(x) = c \cdot (T \circ S)(x) = c \cdot T(S(x)) =$$

$= T(c \cdot S(x))$,

hence $S(x+y) = S(x) + S(y)$ and $S(c \cdot x) = c \cdot S(x)$ because T is injective.

Thus $S: W \longrightarrow V$ is a linear transformation with $S \circ T = id_V$ and $T \circ S = id_W$,

so T is invertible. $\square$.

Corollary 32: Let V and W be vector spaces over the same field $\mathbb{F}$, let $T: V \longrightarrow W$ be a linear transformation, let $\dim(V) = \dim(W)$ be finite. Then T is invertible if and only if $\text{rank}(T) = \dim(W)$.

Proof: Since $\text{rank}(T) = \dim(\text{im}(T))$, we have to prove that T is invertible if and only if $\dim(\text{im}(T)) = \dim(W)$.

($\Rightarrow$) Suppose T is invertible. Then T is surjective by Theorem 31, and $\text{im}(T) = W$ by Theorem 23, so $\dim(\text{im}(T)) = \dim(W)$.

($\Leftarrow$) Suppose $\dim(\text{im}(T)) = \dim(W)$. Since $\dim(V) = \dim(W) = \dim(\text{im}(T))$ then T is injective and surjective by Theorem 24, so T is invertible by Theorem 31. $\square$.

Although an invertible linear transformation $T: V \longrightarrow W$ requires the inverse $T^{-1}: W \longrightarrow V$ to also be a linear transformation, if $T: V \longrightarrow W$ is just injective and surjective then the candidate inverse can be proven to be linear.

Corollary 33: Let V and W be vector spaces over the same field $\mathbb{F}$, let $T: V \longrightarrow W$ be a linear transformation that is injective and surjective. Then the function of sets $S: W \longrightarrow V$ defined by $S(y) = x$ if and only if $T(x) = y$ is the inverse of T.

This was proven for Theorem 31, and essentially states that linearity could be removed from the definition of invertibility. In general, however, this cannot be done.

Theorem 34: Let V and W be vector spaces over the same field $\mathbb{F}$, let $T: V \longrightarrow W$ be an invertible linear transformation. Then V is finite dimensional if and only if W is finite dimensional.

Proof: ($\Rightarrow$) Suppose that $T: V \longrightarrow W$ is invertible, so by Theorem 31 then T is injective and surjective. Suppose that V is finite dimensional, let $\beta = \{v_1, \dots, v_n\}$ be a basis of V. Consider $T(\beta) = \{T(v_1), \dots, T(v_n)\}$, we want to prove that this is a basis of W. Suppose that $T(\beta)$ is not linearly independent, then there are scalars $a_1, \dots, a_n \in \mathbb{F}$, at least one different from zero, such that $a_1 \cdot T(v_1) + \cdots + a_n \cdot T(v_n) = 0$. Then since T is linear $T(a_1 \cdot v_1 + \cdots + a_n \cdot v_n) = 0$. Since T is injective then $a_1 \cdot v_1 + \cdots + a_n \cdot v_n = 0$, which is a linear combination of β with at least one coefficient different from zero, and

thus β is linearly dependent. This contradicts that β is a basis of V, and thus $T(\beta) = \{T(v_1), \dots, T(v_n)\}$ is linearly independent. Moreover since $T(\beta) \subset W$ then $\text{Span}(T(\beta)) \subseteq W$, we now prove $W \subseteq \text{Span}(T(\beta))$. Let $w \in W$, since T is surjective there is a $v \in V$ such that $T(v) = w$. Since β is a basis of V, we can write $v = \sum_{i=1}^{n} a_i \cdot v_i$ for some $a_1, \dots, a_n \in \mathbb{F}$, and thus $w = T(v) = T\left(\sum_{i=1}^{n} a_i \cdot v_i\right) = \sum_{i=1}^{n} a_i \cdot T(v_i)$, so $w \in \text{Span}(T(\beta))$. Hence $W \subseteq \text{Span}(T(\beta))$, so $W = \text{Span}(T(\beta))$. Since $T(\beta)$ is a linearly independent set that generates W, it is a basis of W. Since $T(\beta) = \{T(v_1), \dots, T(v_n)\}$ has the same number of elements as $\beta = \{v_1, \dots, v_n\}$, then the dimension of W coincides with the dimension of V.

($\Leftarrow$) Suppose that $T : V \to W$ is invertible. Then there exists $T^{-1} : W \to V$ the inverse of T. Now by Theorem 31 then T^{-1} is injective and surjective. Suppose that W is finite dimensional, let $\gamma = \{w_1, \dots, w_m\}$ be a basis of W. An analogous reasoning as above proves that $T^{-1}(\gamma) = \{T^{-1}(w_1), \dots, T^{-1}(w_m)\}$ is a basis of V, and indeed the dimension of V coincides with the dimension of W. $\square$.

Theorem 35: Let V and W be finite dimensional vector spaces over the same field $\mathbb{F}$, let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ be basis of V and W respectively, let $T: V \to W$ and $T': V \to W$ be linear transformations such that $[T]_\beta^\gamma = [T']_\beta^\gamma$. Then $T = T'$ as linear transformations.

Proof: Write $T(v_j) = \sum_{i=1}^{m} a_{ij} \cdot w_i$ and $T'(v_j) = \sum_{i=1}^{m} b_{ij} \cdot w_i$ for all $j = 1, \dots, n$, so that $([T]_\beta^\gamma)_{ij} = a_{ij}$ and $([T']_\beta^\gamma)_{ij} = b_{ij}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. Since $[T]_\beta^\gamma = [T']_\beta^\gamma$, then $a_{ij} = ([T]_\beta^\gamma)_{ij} = ([T']_\beta^\gamma)_{ij} = b_{ij}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. Hence $T(v_j) = T'(v_j)$ for all $j = 1, \dots, n$, and thus $T = T'$ by Theorem 25. $\square$.

Theorem 36: Let V and W be finite dimensional vector spaces over the same field $\mathbb{F}$, let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ be basis of V and W respectively, let $T: V \to W$ be a linear transformation. Then:

1) T is invertible if and only if $[T]_\beta^\gamma$ is invertible.

2) If $[T]_\beta^\gamma$ is invertible then $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.

Proof: 1) ($\Rightarrow$) Suppose T is invertible. Then $n = m$ by Theorem 34 and thus $[T]_\beta^\gamma$ is an $n \times n$ matrix. Moreover the inverse $T^{-1}: W \to V$ satisfies $T^{-1}T = \mathrm{id}_V$ and

$TT^{-1} = id_W$. Since $id_V(v_j) = v_j$ and $id_W(w_j) = w_j$ for all $j = 1, \dots, n$, then

$[id_V]_\beta^\beta = I_{dn}$ and $[id_W]_\gamma^\gamma = I_{dn}$ are both the $n \times n$ identity matrix. Using

Theorem 27 then:

$$[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = [T^{-1}T]_\beta^\beta = [id_V]_\beta^\beta = I_{dn},$$

$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [TT^{-1}]_\gamma^\gamma = [id_W]_\gamma^\gamma = I_{dn},$$

so $[T]_\beta^\gamma$ is invertible, with inverse $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.

($\Leftarrow$) Suppose that $[T]_\beta^\gamma$ is invertible. Then there exists a matrix $A \in M_{n \times n}(\mathbb{F})$

such that $[T]_\beta^\gamma A = I_{dn}$ and $A[T]_\beta^\gamma = I_{dn}$. Consider the function

$S: W \longrightarrow V$ defined by $S(w_i) = \sum_{j=1}^{n} A_{ji} \cdot v_j$ for all $i = 1, \dots, n$, where $(A)_{ji} = A_{ji}$

for $i, j = 1, \dots, n$ are the entries of the matrix A. The function S is a well

defined linear transformation by Theorem 25, and by construction $[S]_\gamma^\beta = A$. Now

using Theorem 27 we have:

$$[TS]_\gamma^\gamma = [T]_\beta^\gamma [S]_\gamma^\beta = [T]_\beta^\gamma A = I_{dn} = [id_W]_\gamma^\gamma,$$

$$[ST]_\beta^\beta = [S]_\gamma^\beta [T]_\beta^\gamma = A[T]_\beta^\gamma = I_{dn} = [id_V]_\beta^\beta,$$

and by Theorem 35 then $TS = id_W$ and $ST = id_V$. Hence T is invertible.

2) Suppose $[T]_\beta^\gamma$ is invertible, then by the above T is invertible and:

$$[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = [T^{-1}T]_\beta^\beta = [\mathrm{id}_V]_\beta^\beta = I_{d_n},$$

$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [T T^{-1}]_\gamma^\gamma = [\mathrm{id}_W]_\gamma^\gamma = I_{d_n},$$

so $[T]_\beta^\gamma$ has inverse $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$. □.

Corollary 37: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\beta = \{v_1, \dots, v_n\}$ be a basis of V, let $T: V \to W$ be a linear transformation. Then T is invertible if and only if $[T]_\beta^\beta$ is invertible.

Corollary 38: Let $A \in M_{n\times n}(\mathbb{F})$. Then A is invertible if and only if $T_A: \mathbb{F}^n \to \mathbb{F}^n$, $x \mapsto A\cdot x$ is invertible.

Definition: Let V and W be vector spaces over the field $\mathbb{F}$. We say that V and W are isomorphic if there is an invertible linear transformation $T: V \to W$. Such an invertible linear transformation is called an isomorphism. We denote this by writing $V \cong W$.

Example: Consider the linear transformation $T: \mathbb{F}^n \to \mathbb{F}_{n-1}[x]$ given by $T\left(\begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix}\right) = a_0 + a_1 x + \cdots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1}$. Then T is invertible with inverse $T^{-1}: \mathbb{F}_{n-1}[x] \to \mathbb{F}^n$ given by $T^{-1}(b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}) = \begin{bmatrix} b_0 \\ \vdots \\ b_{n-1} \end{bmatrix}$.

Let $\beta = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\\vdots\\0\end{bmatrix}\right\}$ be the standard basis of $\mathbb{F}^n$, let $\chi = \{1, x, \dots, x^{n-1}\}$

Let $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$ be the standard basis of $\mathbb{F}^n$, let $\beta = 1, x, \ldots, x^{n-1}$

be a basis of $\mathbb{F}_{n-1}[x]$. Recall that we often denote the elements of β by

$e_1, \ldots, e_n$. Then:

$$T(e_1) = 1, \ \ldots, \ T(e_i) = x^{i-1}, \ \ldots, \ T(e_n) = x^{n-1}$$

and thus:

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$ is the $n \times n$ identity matrix.

In particular $\mathbb{F}^n \cong \mathbb{F}_{n-1}[x]$.

Corollary 40: Let V be a vector space over the field $\mathbb{F}$. If V has dimension $n \in \mathbb{N}$

then V is isomorphic to $\mathbb{F}^n$.

Theorem 41: Let V and W be finite dimensional vector spaces over the same field $\mathbb{F}$, let

$\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ be basis of V and W respectively. Then the

function $\Phi: \mathcal{L}(V, W) \longrightarrow M_{m \times n}(\mathbb{F})$ is an isomorphism.

$$T \longmapsto [T]_\beta^\gamma$$

Proof: We first prove that Φ is a linear transformation. Given $T, T' \in \mathcal{L}(V, W)$

and $c \in \mathbb{F}$, by Theorem 26 we have:

$$\Phi(T + T') = [T + T']_\beta^\gamma = [T]_\beta^\gamma + [T']_\beta^\gamma = \Phi(T) + \Phi(T')$$

$$\Phi(c \cdot T) = [c \cdot T]_\beta^\gamma = c \cdot [T]_\beta^\gamma = c \cdot \Phi(T)$$

and thus Φ is a linear transformation. Second, by Theorem 31 it is enough to prove that Φ is injective and surjective to show that it is an isomorphism. For injectivity, notice that if $T \in \ker(\Phi)$ then $[T]_\beta^\gamma = \Phi(T) = 0$ the zero matrix. Thus $T(v_j) = 0$ for all $j = 1, \dots, n$, and thus T is the zero linear transformation, namely $T(v) = 0$ for all $v \in V$, since $Z : V \longrightarrow W$ has $Z(v_j) = T(v_j)$ for all $v \longmapsto 0$ $j = 1, \dots, n$, so $T = Z$ by Theorem 25. This means that $\ker(\Phi) = \{0\}$, so by Theorem 23 then Φ is injective. For surjectivity, let $A \in M_{m \times n}(\mathbb{F})$, and define $S : V \longrightarrow W$ by $S(v_j) = \sum_{i=1}^{m} A_{ij} \cdot w_i$. This is a linear transformation from V to W by Theorem 25, so $S \in \mathcal{L}(V, W)$. Now by construction $A = [S]_\beta^\gamma = \Phi(S)$ and thus Φ is surjective. $\square$.

<u>Theorem 42:</u> Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\beta = \{v_1, \dots, v_n\}$ be a basis of V. Then the function $\phi : V \longrightarrow \mathbb{F}^n$ is an isomorphism.
$v \longmapsto [v]_\beta$

<u>Proof:</u> We first prove that ϕ is a linear transformation. Let $v, v' \in V$ and $c \in \mathbb{F}$, say $[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $[v']_\beta = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ for some $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{F}$. Then:

$$v + v' = \left(\sum_{i=1}^{n} a_i \cdot v_i\right) + \left(\sum_{i=1}^{n} b_i \cdot v_i\right) = \sum_{i=1}^{n} (a_i + b_i) \cdot v_i \text{ , and}$$

$$c \cdot v = c \cdot \left(\sum_{i=1}^{n} a_i \cdot v_i\right) = \sum_{i=1}^{n} (c \cdot a_i) \cdot v_i$$

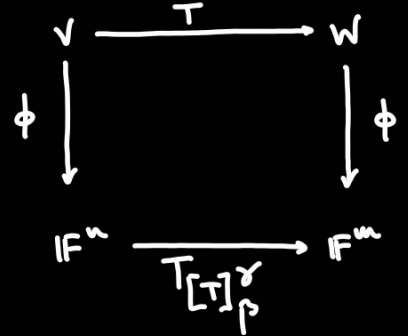
hence:

$$[v+v']_\beta = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [v]_\beta + [v']_\beta, \text{ and}$$

$$[c\cdot v]_\beta = \begin{bmatrix} c\cdot a_1 \\ \vdots \\ c\cdot a_n \end{bmatrix} = c\cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = c\cdot [v]_\beta,$$

and thus ϕ is a linear transformation. Second, by Theorem 31 it is enough to prove that ϕ is injective and surjective to show that it is an isomorphism. For injectivity, notice that if $v \in \ker(\phi)$ then $[v]_\beta = \phi(v) = 0$ the zero vector. Thus $v = 0$ and $\ker(\phi) = \{0\}$, so ϕ is injective by Theorem 23. For surjectivity let $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$ and define $v = \sum_{i=1}^{n} a_i \cdot v_i$, so $v \in V$. Now by construction $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [v]_\beta = \phi(v)$, so ϕ is surjective. □.

We can now restate Theorem 28 as a commutative diagram. Given V and W finite dimensional vector spaces over the same field $\mathbb{F}$, with basis $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ respectively, then:



is a commutative diagram, namely:

$$\phi \circ T = T_{[T]_\beta^\gamma} \circ \phi.$$

Equivalently for all $v \in V$ we have:

$$[T(v)]_\gamma = \phi(T(v)) = \phi \circ T(v) = T_{[T]_\beta^\gamma} \circ \phi(v) = T_{[T]_\beta^\gamma}(\phi(v)) = T_{[T]_\beta^\gamma}([v]_\beta) = [T]_\beta^\gamma \cdot [v]_\beta.$$

When $V = W$, we can use the identity $id_V : V \longrightarrow V$ to change the basis of V.

Definition: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_n\}$ be basis of V. The change of basis matrix from β to γ is $Q = [id_V]_\beta^\gamma$.

Example: Let $V = \mathbb{R}^2$, let $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$. Since:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 5 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then the change of basis matrix from γ to β is:

$$Q = [id_V]_\gamma^\beta = \left[\begin{bmatrix} 2 \\ 3 \end{bmatrix}_\beta \begin{bmatrix} -2 \\ 3 \end{bmatrix}_\beta \right] = \begin{bmatrix} -1 & -5 \\ 3 & 3 \end{bmatrix}.$$

Theorem 43: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_n\}$ be basis of V, let Q be the change of basis matrix from β to γ. Then:

1) Q is invertible with inverse $Q^{-1} = [id_V]_\gamma^\beta$.

2) $[v]_\gamma = Q \cdot [v]_\beta$ for all $v \in V$.

Proof: 1) This is a particular case of Theorem 36.

2) This is a particular case of Theorem 28. □.

When $V=W$, we call linear transformations $T: V \to V$ operators.

Definition: Let V be a vector space over the field $\mathbb{F}$. A linear transformation $T: V \to V$ is called a linear operator on V.

Theorem 44: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_n\}$ be basis of V, let Q be the change of basis matrix from β to γ, let $T: V \to V$ be a linear transformation. Then $[T]_\gamma^\gamma = Q \cdot [T]_\beta^\beta \cdot Q^{-1}$.

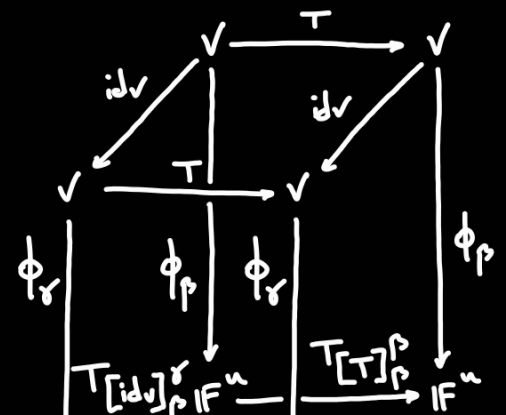
Proof: Note that using Theorem 27 and Theorem 36 we have:

$$Q \cdot [T]_\beta^\beta \cdot Q^{-1} = [id_V]_\beta^\gamma \cdot [T]_\beta^\beta \cdot \left([id_V]_\beta^\gamma\right)^{-1} = [id_V]_\beta^\gamma \cdot [T]_\beta^\beta \cdot [id_V^{-1}]_\gamma^\beta = [id_V]_\beta^\gamma \cdot [T]_\beta^\beta \cdot [id_V]_\gamma^\beta =$$

$$= [id_V \circ T \circ id_V]_\gamma^\gamma = [T]_\gamma^\gamma$$

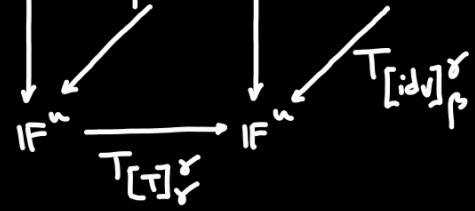
as desired. □.

We can rephrase Theorem 44 as a commutative diagram. Given V a finite dimensional vector space over the field $\mathbb{F}$, with basis $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_n\}$, then:



is a commutative diagram, namely:

$$T_{[id_V]_\beta^\gamma} \circ T_{[T]_\beta^\beta} = T_{[T]_\gamma^\gamma} \circ T_{[id_V]_\beta^\gamma}.$$



Remark: When $V=\mathbb{F}^n$, we can use the standard basis $\sigma=\{e_1,\dots,e_n\}$ to handle every change of basis we may need. Given any other basis $\beta=\{v_1,\dots,v_n\}$ of $\mathbb{F}^n$, then:

$$Q=[id_V]_\beta^\sigma=\left[[id_V(v_1)]_\sigma\ \cdots\ [id_V(v_n)]_\sigma\right]=\left[[v_1]_\sigma\ \cdots\ [v_n]_\sigma\right]=\begin{bmatrix}| & & |\\ v_1 & \cdots & v_n\\ | & & |\end{bmatrix},$$

namely the change of basis matrix from β to σ has the elements of β as columns.

Now given another basis $\gamma=\{w_1,\dots,w_n\}$ of $\mathbb{F}^n$ then by Theorem 27 and Theorem 36:

$$Q=[id_V]_\beta^\gamma=[id_V\circ id_V]_\beta^\gamma=[id_V]_\sigma^\gamma\cdot[id_V]_\beta^\sigma=[id_V^{-1}]_\sigma^\gamma\cdot[id_V]_\beta^\sigma=\left([id_V]_\gamma^\sigma\right)^{-1}\cdot[id_V]_\beta^\sigma=$$

$$=\begin{bmatrix}| & & |\\ w_1 & \cdots & w_n\\ | & & |\end{bmatrix}^{-1}\cdot\begin{bmatrix}| & & |\\ v_1 & \cdots & v_n\\ | & & |\end{bmatrix},$$

namely any change of basis matrix can be obtained via matrices whose columns are the vectors of the basis β and γ.

Example: Let $V=\mathbb{R}^3$ with basis $\sigma=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, $\beta=\left\{\begin{bmatrix}-1\\0\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$, $\gamma=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\10\end{bmatrix}\right\}$.

Since:

$$\begin{bmatrix}-1\\0\\0\end{bmatrix}=\frac{2}{3}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}+\frac{4}{3}\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}+(-1)\cdot\begin{bmatrix}7\\8\\10\end{bmatrix},$$

$$\begin{bmatrix}2\\1\\0\end{bmatrix}=(-2)\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}+1\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}+0\cdot\begin{bmatrix}7\\8\\10\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}=\frac{-1}{3}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}+\frac{1}{3}\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}+0\cdot\begin{bmatrix}7\\8\\10\end{bmatrix},$$

then:

$$[\mathrm{id}_V]_\beta^\gamma = \begin{bmatrix} 2/3 & -2 & -1/3 \\ 4/3 & 1 & 1/3 \\ -1 & 0 & 0 \end{bmatrix}.$$

Also:

$$[\mathrm{id}_V]_\beta^\gamma = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2/3 & -2/3 & 1 \\ -4/3 & 11/3 & -2 \\ 1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -2 & -1/3 \\ 4/3 & 1 & 1/3 \\ -1 & 0 & 0 \end{bmatrix}.$$

Corollary 45: Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{F}^n)$, let $\beta = \{v_1, \dots, v_n\}$ be a basis of $\mathbb{F}^n$, let σ be the standard basis of $\mathbb{F}^n$, let Q be the change of basis matrix from β to σ. Then

$$[T_A]_\beta^\beta = Q^{-1} \cdot A \cdot Q.$$

Proof: This is a particular case of Theorem 44. □.

Definition: Let $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{F})$. We say that A is similar to B if there is $Q \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ invertible such that $B = Q^{-1} A Q$.

Otherwise said, two matrices are similar when they induce the same linear transformation up to a change of basis. We will see that this is an equivalence relation.

4. Determinants.

Definition: A determinant is a function $\det: \mathrm{Mat}_{n\times n}(\mathbb{F}) \longrightarrow \mathbb{F}$ satisfying:

(i) It is linear with respect to each column:

$$\det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} + \det\begin{bmatrix} | & | & & | \\ w_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \det\begin{bmatrix} | & | & & | \\ v_1 + w_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

$$\det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} + \det\begin{bmatrix} | & | & & | \\ v_1 & w_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \det\begin{bmatrix} | & | & & | \\ v_1 & v_2 + w_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

$$\vdots$$

$$\det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} + \det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & w_n \\ | & | & & | \end{bmatrix} = \det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n + w_n \\ | & | & & | \end{bmatrix}$$

$$\det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \cdot \lambda = \det\begin{bmatrix} | & | & & | \\ \lambda v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

$$\det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \cdot \lambda = \det\begin{bmatrix} | & | & & | \\ v_1 & \lambda v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

$$\vdots$$

$$\det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \cdot \lambda = \det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & \lambda v_n \\ | & | & & | \end{bmatrix}$$

(ii) If A has two equal columns, then $\det(A) = 0$.

(iii) $\det(I_n) = 1$.

$$\det\begin{bmatrix} | & & | & | & | \\ c_1, \dots, c_i + c_i', \dots, c_n \\ | & & | & | & | \end{bmatrix} = \det\begin{bmatrix} | & & | & | & | \\ c_1, \dots, c_i, \dots, c_n \\ | & & | & | & | \end{bmatrix} + \det\begin{bmatrix} | & & | & | & | \\ c_1, \dots, c_i', \dots, c_n \\ | & & | & | & | \end{bmatrix}$$

and:

$$\det\begin{bmatrix} | & & | & | & | \\ c_1, \dots, a \cdot c_i, \dots, c_n \\ | & & | & | & | \end{bmatrix} = a \cdot \det\begin{bmatrix} | & & | & | & | \\ c_1, \dots, c_i, \dots, c_n \\ | & & | & | & | \end{bmatrix}.$$

(ii) It is alternating in the columns:

$$\det\begin{bmatrix} | & & | & | & | \\ c_1, \dots, c_i, \dots, c_i, \dots, c_n \\ | & & | & | & | \end{bmatrix} = 0.$$

(iii) The determinant of the identity matrix is 1:

$$\det\begin{bmatrix} | & & | \\ e_1, \dots, e_n \\ | & & | \end{bmatrix} = 1.$$

Theorem 46: Let $n \in \mathbb{N}$, the determinant $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ exists and is unique.

Moreover for $A \in M_{n\times n}(\mathbb{F})$ we have:

(i) Column expansion: $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot A_{ij} \cdot \det(\hat{A}_{ij})$ for all $j = 1, \dots, n$.

(ii) Row expansion: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot A_{ij} \cdot \det(\hat{A}_{ij})$ for all $i = 1, \dots, n$.

Where $\hat{A}_{ij} \in M_{(n-1)\times(n-1)}(\mathbb{F})$ is the matrix obtained from A by removing the i-th row and j-th column. In particular $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

Remark: We have:

1) Let B be the matrix obtained from A by swapping two columns or two rows. Then

$\det(B) = -\det(A)$.

2) Let B be the matrix obtained from A by multiplying a column or row by a scalar $c \in \mathbb{F}$. Then $\det(B) = c \cdot \det(A)$.

3) Let B be the matrix obtained from A by adding to the i-th row or column of A a scalar multiple of the j-th row or column of A, respectively. Then $\det(B) = \det(A)$.

4) If A is a lower or upper triangular matrix then $\det(A) = \prod_{i=1}^{n} A_{ii}$.

5) The matrix A is invertible if and only if $\det(A) \neq 0$.

6) For $A, B \in M_{n\times n}(\mathbb{F})$ then $\det(AB) = \det(A)\cdot\det(B)$, but in general $\det(A+B) \neq \det(A) + \det(B)$.

7) For $A \in M_{n\times n}(\mathbb{F})$ invertible then $\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}$.

8) For $A \in M_{n\times n}(\mathbb{F})$ then $\det(A^t) = \det(A)$.

9) If $A, B \in M_{n\times n}(\mathbb{F})$ are similar matrices then $\det(A) = \det(B)$.

* Small aside on equivalence relations:

Definition: Let A be a set. A *relation* R on A is a subset $R \subseteq A \times A$ of pairs of ordered elements of A. Given R a relation on A, we say that $x, y \in A$ are *related* when $(x,y) \in R$, denoted by $x \sim y$. A relation R on A is called an

equivalence relation when for all $x, y, z \in A$ we have:

(i) Reflexive: $(x, x) \in R$.

(ii) Symmetric: if $(x, y) \in R$ then $(y, x) \in R$.

(iii) Transitive: if $(x, y), (y, z) \in R$ then $(x, z) \in R$.

Examples: 1. Let $A = \mathbb{R}$, then:

$E = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y\}$ is an equivalence relation on $\mathbb{R}$.

$F = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$ is a relation on $\mathbb{R}$.

$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x < y\}$ is a relation on $\mathbb{R}$.

$H = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \geq y\}$ is a relation on $\mathbb{R}$.

$I = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x > y\}$ is a relation on $\mathbb{R}$.

2. Let A be a set, let $f : A \to A$ be a function of sets. Then:

$R = \{(x, y) \in A \times A \mid x \in A, y = f(x)\}$ is a relation on A.

3. Let V be a vector space, let $W \subseteq V$ be a vector subspace of V. Then:

$R = \{(v_1, v_2) \in V \times V \mid v_1 + W = v_2 + W\}$ is an equivalence relation on V.

Namely, being on the same coset on V/W determines an equivalence relation on V.

4. Let $A = M_{n\times n}(\mathbb{F})$ for some $n \in \mathbb{N}$ and some field $\mathbb{F}$. Then:

$$R = \{ (A,B) \in M_{n\times n}(\mathbb{F}) \times M_{n\times n}(\mathbb{F}) \mid A \text{ is similar to } B \}$$

is an equivalence relation on $M_{n\times n}(\mathbb{F})$.

Given two vector spaces, we should think of them as "equivalent" when they are isomorphic.

End of the aside. *

5. Diagonalization:

Given a linear transformation $T: V \to W$ between two vector spaces of the same dimension, we can find a basis β of V and a basis γ of W such that $[T]_\beta^\gamma$ is a diagonal matrix. We will now focus on what happens when $T: V \to V$. Namely, given a finite dimensional vector space V and a linear transformation $T: V \to V$, we want to know when there is a basis β such that $[T]_\beta^\beta$ is a diagonal matrix.

Definition: Let V be a finite dimensional vector space over the field $\mathbb{F}$, a linear transformation $T: V \to V$ is said to be <u>diagonalizable</u> when there is a basis β of V such that $[T]_\beta^\beta$ is diagonal. We say that a square matrix $A \in M_{n\times n}(\mathbb{F})$ is

diagonalizable when the linear transformation $T_A : \mathbb{F}^n \to \mathbb{F}^n$, $x \mapsto Ax$ is diagonalizable.

Note that if $T : V \to V$ is diagonalizable then there is a basis $\beta = \{v_1, \dots, v_n\}$ of V such that $T(v_i) = \lambda_i \cdot v_i$ for some $\lambda_i \in \mathbb{F}$, for all $i = 1, \dots, n$.

Definition: Let V be a vector space over the field $\mathbb{F}$, let $T : V \to V$ be a linear transformation. A vector $v \in V$ is said to be an eigenvector of T when:

(i) v is not zero,

(ii) there exists a scalar $\lambda \in \mathbb{F}$ such that $T(v) = \lambda \cdot v$.

We say that λ is the eigenvalue of T with corresponding eigenvector v.

We can now rephrase the condition of T being diagonalizable in terms of eigenvectors.

Theorem 47: Let V be a finite dimensional vector space over the field $\mathbb{F}$. A linear transformation $T : V \to V$ is diagonalizable if and only if there is a basis $\beta = \{v_1, \dots, v_n\}$ of V such that v_i is an eigenvector of eigenvalue $\lambda_i \in \mathbb{F}$ for all $i = 1, \dots, n$, that is $T(v_i) = \lambda_i \cdot v_i$ for all $i = 1, \dots, n$. If this happens, we have:

$$[T]_\beta^\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $T_A : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Ax$.

Since $T_A(v_1) = v_1$, $T_A(v_2) = v_2$, and $T_A(v_3) = 0$, then v_3 is an eigenvector of

Since $T_A(v_1) = v_1$, $T_A(v_2) = v_2$, and $T_A(v_3) = 0$, then v_1 is an eigenvector of eigenvalue 1, v_2 is an eigenvector of eigenvalue 1, and v_3 is an eigenvector of eigenvalue 0. Since $\{v_1, v_2, v_3\}$ are linearly independent, are inside $\mathbb{R}^3$, and $\dim_{\mathbb{R}}(\mathbb{R}^3) = 3$, then $\beta = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. Hence A is diagonalizable and $[T_A]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

To find the eigenvectors and eigenvalues of a linear transformation $T: V \longrightarrow V$ we first find the eigenvalues $\lambda \in \mathbb{F}$ and we then find the eigenvectors $v \in V$ by solving the equation $T(v) = \lambda \cdot v$.

Theorem 47: Let $A \in M_{n\times n}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. The matrix $A - \lambda \cdot I_n$ is invertible if and only if $(A - \lambda \cdot I_n) \cdot v \neq 0$ for all $v \in \mathbb{F}^n$ such that $v \neq 0$.

Proof: ($\Rightarrow$) Suppose that $A - \lambda \cdot I_n$ is invertible. Let $v \in \mathbb{F}^n$ such that $v \neq 0$. If $(A - \lambda \cdot I_n) \cdot v = 0$ then $v = (A - \lambda \cdot I_n)^{-1} \cdot (A - \lambda \cdot I_n) \cdot v = (A - \lambda \cdot I_n)^{-1} \cdot 0 = 0$, a contradiction. Thus $(A - \lambda \cdot I_n) v \neq 0$, as desired.

($\Leftarrow$) Suppose that $(A - \lambda \cdot I_n) \cdot v \neq 0$ for all non-zero $v \in \mathbb{F}^n$. Consider $T_A: \mathbb{F}^n \to \mathbb{F}^n$, $x \mapsto Ax$, and $v \in \ker(T_A - \lambda \cdot T_{I_n})$. Now:

$$0 = (T_A - \lambda \cdot T_{I_n})(v) = T_A(v) - \lambda \cdot T_{I_n}(v) = Av - \lambda \cdot I_n v = (A - \lambda \cdot I_n) v$$

and thus $v = 0$, since otherwise we would have a contradiction with the hypothesis that $(A - \lambda \cdot I_n) \cdot v \neq 0$ for all non-zero $v \in \mathbb{F}^n$. Hence $\ker(T_A - \lambda \cdot T_{I_n}) = \{0\}$, so $T_A - \lambda \cdot T_{I_n}$ is injective by Theorem 23, so $T_A - \lambda \cdot T_{I_n}$ is surjective by Theorem 24, so $T_A - \lambda \cdot T_{I_n}$ is invertible by Theorem 31. Thus by Theorem 36 then $[T_A - \lambda \cdot I_n]_\sigma^\sigma = [T_A]_\sigma^\sigma - \lambda \cdot [T_{I_n}]_\sigma^\sigma = A - \lambda \cdot I_n$ is invertible. $\square$.

Theorem 48: Let $A \in M_{n \times n}(\mathbb{F})$. A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of $T_A : \mathbb{F}^n \to \mathbb{F}^n$ if and only if $\det(A - \lambda \cdot I_n) = 0$.

Proof: ($\Rightarrow$) Suppose λ is an eigenvalue of $T_A : \mathbb{F}^n \to \mathbb{F}^n$, let $v \in V$ be the non-zero associated eigenvector. Then $\lambda \cdot v = T_A(v) = Av$, and thus $Av - \lambda v = 0$, so $(A - \lambda \cdot I_n)v = 0$. Hence $A - \lambda \cdot I_n$ is not invertible by Theorem 47, and thus $\det(A - \lambda \cdot I_n) = 0$.

($\Leftarrow$) Suppose $\det(A - \lambda \cdot I_n) = 0$, so $A - \lambda \cdot I_n$ is not invertible. Then by Theorem 47 there exists a non-zero vector $v \in \mathbb{F}^n$ such that $(A - \lambda \cdot I_n)v = 0$. Hence $\lambda v = Av = T_A(v)$ and thus v is an eigenvector with associated eigenvalue λ, as desired. $\square$.

This means that eigenvalues are solutions of the equation $\det(A - x \cdot I_n) = 0$, and this is also true for a general linear transformation (as stated in Theorem 50).

Definition: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let β be a basis of V, let $T: V \to V$ be a linear transformation. The characteristic polynomial of T is $p_T(x) = \det\left([T - \lambda \cdot id_V]_\beta^\beta\right)$.

Note that if $\dim_{\mathbb{F}}(V) = n$ then $p_T(x) = \det\left([T]_\beta^\beta - x \cdot [id_V]_\beta^\beta\right) = \det\left([T]_\beta^\beta - x \cdot I_n\right)$.

Note that if $A \in M_{n \times n}(\mathbb{F})$ and $T_A: \mathbb{F}^n \to \mathbb{F}^n$ is left multiplication by A, then
$v \mapsto A \cdot v$
$p_{T_A}(x) = \det(A - x \cdot I_n)$ is often denoted by $p_A(x)$ and said to be the characteristic polynomial of the matrix A. Written like this, the eigenvalues of a linear transformation are exactly the roots of its characteristic polynomial.

Theorem 49: Let $A \in M_{n \times n}(\mathbb{F})$ be an upper triangular matrix. Then the eigenvalues of A are its diagonal entries.

Proof: Since A is upper triangular, then $A - x \cdot I_n$ is also upper triangular, and thus $p_A(x) = \det(A - x \cdot I_n) = (a_{11} - x) \cdots (a_{nn} - x)$. Since the roots of $p_A(x)$ are the diagonal entries $a_{11}, \ldots, a_{nn}$ of A, then these are the eigenvalues of A. $\square$.

Theorem 50: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be

a linear transformation. The scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if it is a root of the characteristic polynomial $p_T(x)$ of T.

Proof: ($\Rightarrow$) Suppose λ is an eigenvalue of T, so there is a non-zero vector $v \in V$ such that $T(v) = \lambda v$. Let β be a basis of V, let $\dim_{\mathbb{F}}(V) = n$. Now:

$$\lambda \cdot [v]_\beta = [\lambda v]_\beta = [T(v)]_\beta = [T]_\beta^\beta [v]_\beta = T_{[T]_\beta^\beta}([v]_\beta) \text{ and } [v]_\beta \neq 0$$

where we have used Theorem 28, and thus λ is an eigenvalue of $T_{[T]_\beta^\beta} : \mathbb{F}^n \to \mathbb{F}^n$.

By Theorem 48 then $0 = \det([T]_\beta^\beta - \lambda \cdot I_n) = p_T(\lambda)$ so λ is a root of $p_T(x)$.

($\Leftarrow$) Let β be a basis of V, let $\dim_{\mathbb{F}}(V) = n$. Suppose λ is a root of $p_T(x)$, so $\det([T]_\beta^\beta - \lambda \cdot I_n) = p_T(\lambda) = 0$. By Theorem 48 then λ is an eigenvalue of $T_{[T]_\beta^\beta} : \mathbb{F}^n \to \mathbb{F}^n$, so there is a non-zero vector $w \in \mathbb{F}^n$ such that $T_{[T]_\beta^\beta}(w) = \lambda w$.

Since $\phi : V \to \mathbb{F}^n$, $v \mapsto [v]_\beta$ is an isomorphism by Theorem 42, it is surjective, and injective by Theorem 31. By surjectivity there is a non zero vector $v \in V$ such that $[v]_\beta = w$. Now:

$$[\lambda v]_\beta = \lambda \cdot [v]_\beta = \lambda w = T_{[T]_\beta^\beta}(w) = [T]_\beta^\beta w = [T]_\beta^\beta [v]_\beta = [T(v)]_\beta$$

where we have used Theorem 28. By injectivity $T(v) = \lambda v$ and thus λ is an eigenvalue of T. $\square$.

We can also characterize eigenvectors of linear transformations.

Theorem 50: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation. A vector $v \in V$ is an eigenvector if and only if v is not zero and $v \in \ker(T - \lambda \cdot id_V)$ for some scalar $\lambda \in \mathbb{F}$.

Proof: ($\Rightarrow$) Suppose $v \in V$ is an eigenvector of T of eigenvalue λ, so $T(v) = \lambda v$ and $v \neq 0$. Also $(T - \lambda \cdot id_V)(v) = 0$ so $v \in \ker(T - \lambda \cdot id_V)$.

($\Leftarrow$) Suppose $v \in \ker(T - \lambda \cdot id_V)$ for some $\lambda \in \mathbb{F}$ and $v \neq 0$. Then $(T - \lambda \cdot id_V)(v) = 0$ so $T(v) = \lambda v$ and v is an eigenvector of eigenvalue λ. $\square$.

Hence all eigenvectors of eigenvalue λ are in the vector subspace $\ker(T - \lambda \cdot id_V)$ of V.

Definition: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\lambda \in \mathbb{F}$, let $T: V \to V$ be a linear transformation. The vector subspace $\ker(T - \lambda \cdot id_V)$ of V is called the eigenspace of λ, and is denoted by E_λ.

Given $T: V \to V$ a linear transformation and $\lambda \in \mathbb{F}$, the eigenspaces of λ satisfy $T(E_\lambda) \subseteq E_\lambda$. To see this, let $v \in E_\lambda$, so $T(v) = \lambda v$. Now:

$$(T - \lambda \cdot id_V)(T(v)) = T(T(v)) - \lambda \cdot T(v) = T(\lambda v) - T(\lambda v) = 0$$

and thus $T(v) \in E_\lambda$ for all $v \in E_\lambda$.

Example: Find the real eigenvalues and eigenspaces of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. We first compute the characteristic polynomial:

$$p_A(x) = \det(A - x \cdot I_2) = \det \begin{bmatrix} 1-x & 2 \\ 3 & 2-x \end{bmatrix} = (1-x)(2-x) - 6 = x^2 - 3x - 4 = (x-4)(x+1)$$

and thus $\lambda = -1$ and $\lambda = 4$ are the eigenvalues of A.

To compute E_4 we want to find the vectors $v \in \mathbb{R}^2$ such that $(A - 4 \cdot I_2)v = 0$:

$$\begin{bmatrix} -3 & 2 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{yields} \quad \begin{matrix} -3x + 2y = 0 \\ 3x - 2y = 0 \end{matrix} \quad \text{namely} \quad 3x = 2y.$$

Let $t \in \mathbb{R}$ be a free variable, set $x = t$, then $y = \frac{3}{2}t$ and $v = \begin{bmatrix} t \\ \frac{3}{2}t \end{bmatrix}$ so:

$$E_\lambda = \ker(A - 4 \cdot I_2) = \left\{ \begin{bmatrix} t \\ \frac{3}{2}t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right).$$

To compute E_{-1} we want to find the vectors $v \in \mathbb{R}^2$ such that $(A + I_2)v = 0$:

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{yields} \quad \begin{matrix} 2x + 2y = 0 \\ 3x + 3y = 0 \end{matrix} \quad \text{namely} \quad x = -y.$$

Let $t \in \mathbb{R}$ be a free variable, set $x = t$, then $y = -t$ and $v = \begin{bmatrix} t \\ -t \end{bmatrix}$ so:

$$E_\lambda = \ker(A + I_2) = \left\{ \begin{bmatrix} t \\ -t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right).$$

Unfortunately, not all linear transformations have eigenvalues.

Example: Find the real eigenvalues and eigenspaces of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, which is the rotation of $\mathbb{R}^2$ by an angle of $\theta \in [0, 2\pi)$ counterclockwise. We first compute the characteristic polynomial:

$$p_A(x) = \det(A - x \cdot I_2) = \det\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} = (\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta =$$

$$= \cos^2\theta - 2\cos\theta \cdot x + x^2 + \sin^2\theta = x^2 - 2\cos\theta \cdot x + 1.$$

This polynomial has discriminant $\sqrt{4 \cdot \cos^2\theta - 4} \notin \mathbb{R}$, and thus $p_A(x)$ has no real roots. Hence A has no eigenvalues, and thus no eigenvectors. All the eigenspaces are exactly zero.

Theorem 51: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation, let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. Then the corresponding eigenvectors $v_1, \ldots, v_k$ are linearly independent.

Proof: We proceed by induction on k the number of eigenvalues.

For $n = 1$ we have a single eigenvalue λ_1 and eigenvector v_1. Since $v_1 \neq 0$ the set $\{v_1\}$ is linearly independent.

Suppose the statement is true for $n = k-1$. Namely, suppose that if $\lambda_1, \ldots, \lambda_{k-1}$ are distinct eigenvalues of T, then their corresponding eigenvectors $v_1, \ldots, v_{k-1}$ are linearly

independent.

Suppose $n=k$, that is, we have $\lambda_1,\dots,\lambda_k$ distinct eigenvalues of T with $v_1,\dots,v_k$ the corresponding eigenvectors. Suppose that there are scalars $a_1,\dots,a_k\in\mathbb{F}$ such that:

$$a_1v_1+\cdots+a_kv_k=0.$$

Then applying $T-\lambda_k\cdot id_V$ to the above equality yields:

$$(T-\lambda_k\cdot id_V)(a_1v_1+\cdots+a_kv_k)=(T-\lambda\cdot id_V)(0)=0,$$

$$(T-\lambda_k\cdot id_V)(a_1v_1)+\cdots+(T-\lambda_k\cdot id_V)(a_{k-1}\cdot v_{k-1})+(T-\lambda_k\cdot id_V)(a_kv_k)=0,$$

$$T(a_1v_1)-\lambda_ka_1v_1+\cdots+T(a_{k-1}v_{k-1})-\lambda_ka_{k-1}v_{k-1}+T(a_kv_k)-\lambda_ka_kv_k=0,$$

$$a_1T(v_1)-\lambda_ka_1v_1+\cdots+a_{k-1}T(v_{k-1})-\lambda_ka_{k-1}v_{k-1}+a_kT(v_k)-\lambda_ka_kv_k=0,$$

$$a_1\lambda_1v_1-\lambda_ka_1v_1+\cdots+a_{k-1}\lambda_{k-1}v_{k-1}-\lambda_ka_{k-1}v_{k-1}+a_k\lambda_kv_k-\lambda_ka_kv_k=0,$$

$$a_1(\lambda_1-\lambda_k)v_1+\cdots+a_k(\lambda_{k-1}-\lambda_k)v_{k-1}=0,$$

a linear combination of the eigenvectors $v_1,\dots,v_{k-1}$. Since $\lambda_1,\dots,\lambda_{k-1}$ are the distinct associated eigenvalues, by induction hypothesis their respective eigenvectors $v_1,\dots,v_{k-1}$ are linearly independent. Hence:

$$a_1(\lambda_1-\lambda_k)=0,\ \dots,\ a_{k-1}(\lambda_{k-1}-\lambda_k)=0$$

and since all the $\lambda_1,\dots,\lambda_k$ are distinct then $(\lambda_i-\lambda_k)\neq0$ for all $i=1,\dots,k-1$ so:

$a_1 = 0, \dots, a_{k-1} = 0$

and thus our linear combination is $a_k v_k = 0$, so $a_k = 0$ since $v_k \neq 0$ because it is an eigenvector. Since the only way of having a linear combination of $v_1, \dots, v_k$ equal zero is by having all coefficients be zero, the vectors $v_1, \dots, v_k$ are linearly independent. □.

Corollary 52: Let V be a vector space over the field $\mathbb{F}$, let $\dim_{\mathbb{F}}(V) = n \in \mathbb{N}$, let $T: V \to V$ be a linear transformation with n distinct eigenvalues. Then T is diagonalizable.

Proof: Let $\lambda_1, \dots, \lambda_n$ be distinct eigenvalues of T. The set $\beta = \{v_1, \dots, v_n\}$ is linearly independent by Theorem 51, and thus $\text{span}(\beta) \subseteq V$ is a vector subspace of dimension n. By Theorem 17 then $\text{span}(\beta) = V$ so β is a basis of V composed of eigenvectors of V, so by Theorem 47 then T is diagonalizable. □.

The converse of Corollary 51 is not true: the identity function $id_V: V \to V$ is diagonalizable but it only has 1 as eigenvalue, so if $\dim_{\mathbb{F}}(V) > 1$ we have a counterexample. In general, a diagonal matrix can have repeated entries in the diagonal.

Definition: A polynomial $p(x) \in \mathbb{F}[x]$, $\deg(p(x)) = n \in \mathbb{N}$, is said to split over $\mathbb{F}$ when

there are scalars $a_1, \dots, a_n, c \in \mathbb{F}$ such that $p(x) = c \cdot (x-a_1) \cdots (x-a_n)$.

Equivalently, a polynomial splits over $\mathbb{F}$ if it can be completely factored into linear terms in $\mathbb{F}[x]$, or equivalently if it has as many roots in $\mathbb{F}$ as its degree.

Example: The polynomial x^2+1 is not split over $\mathbb{R}$, but since $x^2+1 = (x-i)(x+i)$ it is split over $\mathbb{C}$.

Theorem 52: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation. If T is diagonalizable then $p_T(x)$ the characteristic polynomial of T is split over $\mathbb{F}$.

Proof: Suppose that T is diagonalizable, so there exists a basis β of V such that $[T]_\beta^\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is a diagonal matrix with entries $\lambda_1, \dots, \lambda_n \in \mathbb{F}$, not necessarily distinct, and $\dim_{\mathbb{F}}(V) = n \in \mathbb{N}$. The characteristic polynomial of T is:

$$p_T(x) = \det\left([T]_\beta^\beta - x \cdot I_n\right) = \det \begin{bmatrix} \lambda_1 - x & & 0 \\ & \ddots & \\ 0 & & \lambda_n - x \end{bmatrix} = (\lambda_1 - x) \cdots (\lambda_n - x)$$

and thus $p_T(x)$ splits over $\mathbb{F}$. $\square$.

Remark: The converse of Theorem 52 is false: there are linear transformations $T: V \to V$ that are not diagonalizable but whose characteristic polynomial $p_T(x)$ splits. Let:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{consider} \quad T_A: \mathbb{R}^2 \to \mathbb{R}^2. \qquad v \mapsto Av$$

Now $p_{T_A}(x) = \det(A - x \cdot I_2) = \det\begin{bmatrix} 1-x & 1 \\ 0 & 1-x \end{bmatrix} = (1-x)(1-x)$ and thus $p_T(x)$ splits over $\mathbb{R}$. Suppose that T_A is diagonalizable, so there exists a basis β of $\mathbb{R}^2$ such that $[T]_\beta^\beta = D$ is a diagonal matrix. We know that A and D are similar, so if $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ then $p_T(x) = \det(D - x \cdot I_2) = \det\begin{bmatrix} d_1 - x & 0 \\ 0 & d_2 - x \end{bmatrix} = (d_1 - x)(d_2 - x)$. Since $p_T(x)$ has roots d_1, d_2, and 1, then $d_1 = 1 = d_2$, so $D = I_2$. Since A and I_2 are similar then there is an invertible matrix $Q \in M_{2\times 2}(\mathbb{R})$ such that $A = Q^{-1} I_2 Q$.

But now $A = Q^{-1} I_2 Q = Q^{-1} Q = I_2$, which is a contradiction since $A \neq I_2$.

Hence T_A is not diagonalizable.

Since the characteristic polynomial $p_T(x)$ of a diagonalizable operator T is split, we want to know its roots (because these will be the eigenvalues) and how many times each root appears (namely, the multiplicity of each root).

Definition: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\lambda \in \mathbb{F}$, let $T: V \to V$ be a linear transformation. The algebraic multiplicity of λ in T is the largest positive integer k such that $(x-\lambda)^k$ divides $p_T(x)$, which we denote by $\text{mult}_T(\lambda)$ or just m_λ if the linear transformation T is understood.

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$, then $p_A(x) = (x-1)^2 \cdot (-x)$ and thus the eigenvalues of A are 1 and 0 with algebraic multiplicities 2 and 1 respectively.

Note that the sum of the algebraic multiplicities of a linear transformation $T: V \to V$ is less than or equal to the dimension of V, with equality exactly when $p_T(x)$ is split over $\mathbb{F}$.

Definition: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\lambda \in \mathbb{F}$, let $T: V \to V$ be a linear transformation. The geometric multiplicity of λ in T is the dimension of the vector space $E_\lambda = \ker(T - \lambda \cdot id_V)$, which we denote by $gmult_T(\lambda)$.

Example: Let $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$, then the eigenvalues of A are 1 and 0 with eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ respectively. Then $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$ so $gmult_T(1) = 2$, and $E_0 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$ so $gmult_T(0) = 1$.

In the case above we saw that the geometric multiplicities coincided with the algebraic multiplicities. In general, this is not true, this will only be true when the linear operator is diagonalizable. What is true is that the geometric multiplicity is always bounded above by the algebraic multiplicity.

Theorem 53: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be

a linear transformation, let $\lambda \in \mathbb{F}$ be an eigenvalue of T. Then $1 \leq \text{gmult}_T(\lambda) \leq \text{mult}_T(\lambda)$.

Proof: Since $\lambda \in \mathbb{F}$ is an eigenvalue of T, there is an associated eigenvector $v \in V$, and thus E_λ contains v. Since E_λ is a vector space, it contains $\text{span}(v)$ the scalar multiples of v, and thus $\text{gmult}_T(\lambda) = \dim_{\mathbb{F}}(E_\lambda) \geq \dim(\text{span}(v)) = 1$ by Theorem 17.

Let $k = \dim_{\mathbb{F}}(E_\lambda) = \text{gmult}_T(\lambda)$ and $\gamma = \{v_1, \dots, v_k\}$ be a basis of E_λ. Extend γ to a basis $\beta = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V by Corollary 18, say $n = \dim_{\mathbb{F}}(V)$. Now:

$$[T]_\beta^\beta = \left[\begin{array}{ccc|c} \lambda & & 0 & \\ & \ddots & & B \\ 0 & & \lambda & \\ \hline & 0 & & C \end{array}\right] = \left[\begin{array}{c|c} \lambda \cdot I_k & B \\ \hline 0 & C \end{array}\right]$$

where $B \in M_{k \times (n-k)}(\mathbb{F})$, $C \in M_{(n-k)\times(n-k)}(\mathbb{F})$, and $0 \in M_{k \times k}(\mathbb{F})$. Hence:

$$[T]_\beta^\beta - x \cdot I_n = \left[\begin{array}{ccc|c} \lambda - x & & 0 & \\ & \ddots & & B \\ 0 & & \lambda - x & \\ \hline & 0 & & C - x \cdot I_{(n-k)\times(n-k)} \end{array}\right] = \left[\begin{array}{c|c} (\lambda - x) \cdot I_k & B \\ \hline 0 & C - x \cdot I_{(n-k)\times(n-k)} \end{array}\right]$$

and we have:

$$p_T(x) = \det\left([T]_\beta^\beta - x \cdot I_n\right) = \det\left(\left[\begin{array}{c|c} (\lambda - x) \cdot I_k & B \\ \hline 0 & C - x \cdot I_{(n-k)\times(n-k)} \end{array}\right]\right) =$$

$$= \det\left((\lambda - x) \cdot I_k\right) \cdot \det\left(C - x \cdot I_{(n-k)\times(n-k)}\right) = (\lambda - x)^k \cdot g(x)$$

where $g(x) = \det\left(C - x \cdot I_{(n-k)\times(n-k)}\right)$ is a polynomial of degree $n-k$. This means that $(\lambda - x)^k$ divides $p_T(x)$, and since $m_\lambda = \text{mult}_T(\lambda)$ is the largest integer such that $(\lambda - x)^{m_\lambda}$ divides $p_T(x)$, then $k \leq m_\lambda$. Putting all the above together we obtain:

$1 \leq \text{gmult}_T(\lambda) = k \leq m_\lambda = \text{mult}_T(\lambda)$, namely $1 \leq \text{gmult}_T(\lambda) \leq \text{mult}_T(\lambda)$. $\square$.

Example: Let $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$, consider $T_A : \mathbb{R}^3 \to \mathbb{R}^3$, $v \mapsto Av$. Now $E_1 \oplus E_0$ is a vector subspace of $\mathbb{R}^3$ and $\dim_{\mathbb{R}}(E_1 \oplus E_0) = 3 = \dim_{\mathbb{R}}(\mathbb{R}^3)$, and thus $\mathbb{R}^3 = E_1 \oplus E_0$. In particular A is diagonalizable since $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ yields $[T_A]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Moreover, comparing with the standard basis $\sigma = \{e_1, e_2, e_3\}$ gives:

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = A = [T_A]_\sigma^\sigma = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = [\text{id}_V]_\beta^\sigma [T_A]_\beta^\beta ([\text{id}_V]_\beta^\sigma)^{-1}.$$

Example: Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, then $p_A(x) = (1-x)^2(3-x)$. Hence A has eigenvalues 1 and 3, with algebraic multiplicities 2 and 1 respectively, and eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. In particular; $p_A(x)$ splits over $\mathbb{R}$, $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$, $E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right)$, and thus the geometric multiplicities of 1 and 3 are 1 and 1 respectively. Since we can find at most two linearly independent eigenvectors, one in E_1 and the other in E_2, and the dimension of $\mathbb{R}^3$ is three, we cannot find a basis $\beta = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ such that v_1, v_2, v_3 are linearly independent eigenvectors of $T_A : \mathbb{R}^3 \to \mathbb{R}^3$, $v \mapsto Av$. Hence T_A is not a diagonalizable linear transformation.

A slightly more general version of Theorem 51 will be useful.

Theorem 54: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be distinct eigenvalues of T. Let $S_1 \subset E_1, \dots, S_k \subset E_k$ be linearly independent subsets of the eigenspaces of $\lambda_1, \dots, \lambda_k$ respectively. Then $S_1 \cup \dots \cup S_k$ is a linearly independent subset of V.

Proof: We proceed by induction on n the number of distinct eigenvalues.

For $n=1$ we have $S_1 \subset E_1$ a linearly independent subset of E_1. Then E_1 is a linearly independent subset of V, as desired.

Suppose that the statement is true for $n=k-1$, namely given $S_1 \subset E_1, \dots, S_{k-1} \subset E_{k-1}$ linearly independent subsets of the distinct eigenspaces of $\lambda_1, \dots, \lambda_{k-1}$ respectively then $S_1 \cup \dots \cup S_{k-1}$ is a linearly independent subset of V.

We now prove the case $n=k$. We have $S_1 \subset E_1, \dots, S_k \subset E_k$ linearly independent subsets of the eigenspaces of $\lambda_1, \dots, \lambda_k$ respectively, so in particular $S_1 \subset E_1, \dots, S_{k-1} \subset E_{k-1}$ are linearly independent subsets of the eigenspaces of $\lambda_1, \dots, \lambda_{k-1}$ respectively. Hence by induction hypothesis the subset $S_1 \cup \dots \cup S_{k-1}$ of V is linearly independent. Consider now a linear combination of $S_1 \cup \dots \cup S_k$, namely let $S_1 = \{v_{1,1}, \dots, v_{1,n_1}\}, \dots, S_k = \{v_{k,1}, \dots, v_{k,n_k}\}$ and let

$a_{1,1}, \dots, a_{1,n_1}, \dots, a_{k,1}, \dots, a_{k,n_k} \in \mathbb{F}$ be such that:

$$a_{1,1} v_{1,1} + \dots + a_{1,n_1} v_{1,n_1} + \dots + a_{k,1} v_{k,1} + \dots + a_{k,n_k} v_{k,n_k} = 0.$$

Applying the linear transformation $T - \lambda_k \cdot id_V$ we obtain:

$$(\lambda_1 - \lambda_k) a_{1,1} v_{1,1} + \dots + (\lambda_1 - \lambda_k) a_{1,n_1} v_{1,n_1} + \dots +$$

$$(\lambda_{k-1} - \lambda_k) a_{k-1,1} v_{k-1,1} + \dots + (\lambda_{k-1} - \lambda_k) a_{k-1,n_{k-1}} v_{k-1,n_{k-1}} = 0$$

since $(T - \lambda_k \cdot id_V)(a_{k,1} v_{k,1}) = \dots = (T - \lambda_k \cdot id_V)(a_{k,n_k} v_{k,n_k}) = 0$. This is a linear combination of elements in $S_1 \cup \dots \cup S_{k-1}$ equal to zero, and thus by linear independence of $S_1 \cup \dots \cup S_{k-1}$ we have:

$$(\lambda_1 - \lambda_k) a_{1,1} = 0, \dots, (\lambda_1 - \lambda_k) a_{1,n_1} = 0, \dots, (\lambda_{k-1} - \lambda_k) a_{k-1,1} = 0, \dots, (\lambda_{k-1} - \lambda_k) a_{k-1,n_{k-1}} = 0$$

Since $\lambda_1, \dots, \lambda_k$ are distinct we in fact have:

$$a_{1,1} = 0, \dots, a_{1,n_1} = 0, \dots, a_{k-1,1} = 0, \dots, a_{k-1,n_{k-1}} = 0$$

and thus our starting linear combination becomes:

$$a_{k,1} v_{k,1} + \dots + a_{k,n_k} v_{k,n_k} = 0.$$

Since S_k is a linearly independent subset of E_k then we have $a_{k,1} = 0, \dots, a_{k,n_k} = 0$.

Hence all the coefficients of our starting linear combination are zero, and thus the set

$S_1 \cup \cdots \cup S_k$ is linearly independent. $\square$.

Alternatively, note that $a_{1,1}v_{1,1} + \cdots + a_{1,n_1}v_{1,n_1} \in E_1, \ldots, a_{k,1}v_{k,1} + \cdots + a_{k,n_k}v_{k,n_k} \in E_k$ are k eigenvectors having distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Hence by Theorem 51 they are linearly independent. If there exists a linear combination:

$$a_{1,1}v_{1,1} + \cdots + a_{1,n_1}v_{1,n_1} + \cdots + a_{k,1}v_{k,1} + \cdots + a_{k,n_k}v_{k,n_k} = 0$$

where some of the coefficients are non-zero, this contradicts the aforementioned linear independence, and thus this cannot happen. Here we are using that $a_{1,1}v_{1,1} + \cdots + a_{1,n_1}v_{1,n_1} \neq 0, \ldots,$ $a_{k,1}v_{k,1} + \cdots + a_{k,n_k}v_{k,n_k} \neq 0$ when among $a_{1,1}, \ldots, a_{1,n_1}, \ldots, a_{k,1}, \ldots, a_{k,n_k} \in \mathbb{F}$ there are at least a non-zero $a_{i,j}$ for all $i = 1, \ldots, k$.

We only need one more result to completely characterize diagonalizable operators.

Theorem 55: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation with $p_T(x)$ split over $\mathbb{F}$, let $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ be the distinct eigenvalues of T. Then:

1) T is diagonalizable if and only if $\text{mult}_T(\lambda_i) = \text{gmult}_T(\lambda_i)$ for all $i = 1, \ldots, k$.

2) If T is diagonalizable and β_i is a basis of the eigenspace E_i with corresponding eigenvector λ_i for all $i = 1, \ldots, k$, then $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis of eigenvectors of V.

Proof: Set $n = \dim_{\mathbb{F}}(V)$, $d_i = \text{gmult}_T(\lambda_i) = \dim_{\mathbb{F}}(E_i)$, and $m_i = \text{mult}_T(\lambda_i)$.

1) ($\Rightarrow$) Suppose that T is diagonalizable, so there is a basis β of V that is formed by eigenvectors of T. Let $\beta_i = \beta \cap E_i$ and set $u_i = |\beta_i| = \dim_{\mathbb{F}}(\text{span}(\beta_i))$ for $i = 1, \ldots, k$. Since the elements of β are eigenvectors, each of them is in exactly one eigenspace, and thus $\beta = \beta_1 \cup \cdots \cup \beta_k$ with $n = |\beta| = |\beta_1| + \cdots + |\beta_k| = u_1 + \cdots + u_k$. Also, note that $\text{span}(\beta_i) \subseteq E_i$ and thus $u_i \leq d_i$ for all $i = 1, \ldots, k$, and also $d_i \leq m_i$ by Theorem 53. Moreover, since $p_T(x)$ is split over $\mathbb{F}$ then $p_T(x) = (\lambda_1 - x)^{m_1} \cdots (\lambda_k - x)^{m_k}$ and since $p_T(x)$ has degree n the dimension of V then $n = \deg(p_T(x)) = m_1 + \cdots + m_k$. Hence:

$$u_1 + \cdots + u_k = n = m_1 + \cdots + m_k \quad \text{and thus} \quad (m_1 - u_1) + \cdots + (m_k - u_k) = 0$$

with $m_i - u_i \geq 0$ for all $i = 1, \ldots, k$, whence $m_i - u_i = 0$ for all $i = 1, \ldots, k$. Thus:

$$u_i \leq d_i \leq m_i \quad \text{and} \quad u_i = m_i \text{ for all } i = 1, \ldots, k,$$

so $\text{gmult}_T(\lambda_i) = d_i = m_i = \text{mult}_T(\lambda_i)$ for all $i = 1, \ldots, k$.

($\Leftarrow$) Suppose that $\text{gmult}_T(\lambda_i) = d_i = m_i = \text{mult}_T(\lambda_i)$ for all $i = 1, \ldots, k$, let β_i be a basis of E_i the eigenspace corresponding to λ_i for all $i = 1, \ldots, k$. Now $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a linearly independent subset of V by Theorem 54, and since eigenvectors have a unique associated eigenvalue then $\beta_i \cap \beta_j = \emptyset$ when $i \neq j$, whence:

$$|\beta| = |\beta_1| + \cdots + |\beta_k| = d_1 + \cdots + d_k = m_1 + \cdots + m_k = p_T(x) = n$$

because $p_T(x)$ splits as $p_T(x) = (\lambda_1 - x)^{m_1} \cdots (\lambda_k - x)^{m_k}$ over $\mathbb{F}$. Hence $\text{span}(\beta) \subseteq V$ and $\dim_{\mathbb{F}}(\text{span}(\beta)) = |\beta| = n$, so by Theorem 17 then $\text{span}(\beta) = V$ and β is a basis of V formed by eigenvalues of T, meaning that T is diagonalizable.

2) Suppose that T is diagonalizable, then by the above $\text{gmult}_T(\lambda_i) = d_i = m_i = \text{mult}_T(\lambda_i)$ for all $i = 1, \ldots, k$. Given $\beta_1, \ldots, \beta_k$ basis of the eigenspaces $E_1, \ldots, E_k$ respectively, then by the proof above $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis of V formed by eigenvectors of T, as desired. □.

We can now fully characterize diagonalizable operators.

Theorem 56: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation. Then T is diagonalizable if and only if $p_T(x)$ is split over $\mathbb{F}$ and $\text{mult}_T(\lambda) = \text{gmult}_T(\lambda)$ for all eigenvalues $\lambda \in \mathbb{F}$ of T.

Proof: ($\Rightarrow$) Suppose that T is diagonalizable. Then $p_T(x)$ is split over $\mathbb{F}$ by Theorem 52, and thus $\text{mult}_T(\lambda) = \text{gmult}_T(\lambda)$ for all eigenvalues $\lambda \in \mathbb{F}$ of T by Theorem 55.

($\Leftarrow$) Suppose that $p_T(x)$ is split over $\mathbb{F}$ and that $\text{mult}_T(\lambda) = \text{gmult}_T(\lambda)$ for all eigenvalues

$\lambda \in \mathbb{F}$ of T. Then T diagonalizes by Theorem 55. $\square$

This characterization can be rewritten in terms of a direct sum.

Definition: Let V be a vector space over the field $\mathbb{F}$, let $W_1, \dots, W_k$ be vector subspaces of V.

We say that V is the direct sum of $W_1, \dots, W_k$, denoted $V = W_1 \oplus \cdots \oplus W_k$, when:

1) $V = W_1 + \cdots + W_k$, and

2) $W_i \cap (W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_k) = \{0\}$ for all $i = 1, \dots, k$.

The above is a generalization of the direct sum of two vector subspaces. As such, there are analogous ways of characterizing it, mimicking the case of two vector subspaces.

Theorem 57: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $W_1, \dots, W_k$ be vector subspaces of V. The following are equivalent:

(1) $V = W_1 \oplus \cdots \oplus W_k$,

(2) $V = W_1 + \cdots + W_k$ and if $w_1 + \cdots + w_k = 0$ with $w_1 \in W_1, \dots, w_k \in W_k$ then

$w_1 = \cdots = w_k = 0$.

(3) For all $v \in V$ there is a unique decomposition $v = w_1 + \cdots + w_k$ with $w_i \in W_i$

for $i = 1, \dots, k$.

(4) If $\beta_1, \dots, \beta_k$ are ordered basis of $W_1, \dots, W_k$ respectively, then $\beta = \beta_1 \cup \cdots \cup \beta_k$

is an ordered basis of V.

(5) There exists ordered basis $\beta_1, \dots, \beta_k$ of $W_1, \dots, W_k$ respectively such that $\beta = \beta_1 \cup \cdots \cup \beta_k$ is an ordered basis of V.

Proof: First, prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) for the case $k=2$. Then use induction to prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1). $\square$.

Theorem 58: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation. Then T is diagonalizable if and only if V is the direct sum of the eigenspaces of T. Namely, set $E_1, \dots, E_k$ the eigenspaces of T, then T is diagonalizable if and only if $V = E_1 \oplus \cdots \oplus E_k$.

Proof: ($\Rightarrow$) Suppose that T is diagonalizable, let β_i be a basis of the eigenspace E_i with corresponding eigenvector λ_i for all $i=1, \dots, k$. Then $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis of eigenvectors of V by Theorem 55, and thus $V = E_1 \oplus \cdots \oplus E_k$ by Theorem 57 (4).

($\Leftarrow$) Suppose that $V = E_1 \oplus \cdots \oplus E_k$ where E_i is the eigenspace corresponding to the eigenvalue λ_i for all $i=1, \dots, k$. Then by Theorem 57(5) there exist basis $\beta_1, \dots, \beta_k$ of $E_1, \dots, E_k$ respectively such that $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis of V. The basis β is formed by eigenvectors of T, and thus T is diagonalizable. $\square$.

The motivating questions for diagonalization have been:

(a) Let $T \in \mathcal{L}(V,W)$, can we find a basis β of V and γ of W such that $[T]_\beta^\gamma$ is a diagonal matrix?

(b) Let $T \in \mathcal{L}(V,V)$, can we find a basis β of V such that $[T]_\beta^\beta$ is a diagonal matrix?

The next natural question is, given $T, S \in \mathcal{L}(V,V)$, can we find a basis β of V such that $[T]_\beta^\beta$ and $[S]_\beta^\beta$ are diagonal matrices?

Definition: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ and $S: V \to V$ be linear transformations. We say that T and S are simultaneously diagonalizable when there is a basis β of V such that $[T]_\beta^\beta$ and $[S]_\beta^\beta$ are diagonal matrices.

Theorem 59: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ and $S: V \to V$ be linear transformations. Then T and S are simultaneously diagonalizable if and only if T is diagonalizable, S is diagonalizable, and $ST = TS$ in $\mathcal{L}(V)$.

Proof: Exercise. (hard, but conceptually very important!) □.

Because of Theorem 59 there is not much more to do with diagonalization. Instead,

note that given $T \in \mathcal{L}(V)$ and $p_T(x) \in \mathbb{F}[x]$, we can form $p_T(T) \in \mathcal{L}(V)$. We will now turn to understand $p_T(T)$.

Remark: Given a polynomial $f(x) \in \mathbb{F}[x]$, say $f(x) = a_n x^n + \cdots + a_1 x + a_0$ for some $a_0, \ldots, a_n \in \mathbb{F}$, and $T \in \mathcal{L}(V)$, then $f(T) = a_n T^n + \cdots + a_1 T + a_0$ is a linear transformation from V to V given by $f(T): V \longrightarrow V$.

$$v \mapsto a_n T^n(v) + \cdots + a_1 T(v) + a_0 \cdot v$$

Since $f(T)$ is formed by composing linear transformations, multiplying linear transformations by a scalar, and adding linear transformations, $f(T)$ is a linear transformation.

Definition: Let V be a vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation. A vector subspace $W \subseteq V$ is said to be T-invariant when $T(W) \subseteq W$. Given $v \in V$, the T-cyclic vector subspace of V generated by v, denoted W_v, is the span of the successive applications of T to v, namely:

$$W_v = \text{span}\{v, T(v), T^2(v), \ldots\} = \text{span}\{T^i(v) \mid i \in \mathbb{N}\}.$$

Remark: The T-cyclic vector subspace of V generated by v is the smallest T-invariant subspace of V containing v. First, W_v is T-invariant because given $w \in W$ then

$w = T^i(v)$ for some $i \in \mathbb{N}$, whence $T(w) = T^{i+1}(v) \in W_v$ and thus $T(W_v) \subseteq W_v$.

Second, if W is a T-invariant subspace of V containing v, then not only $v \in W$ but also $T(v) \in T(W) \subseteq W$, $T^2(v) \in T(T(W)) \subseteq T(W) \subseteq W$, and in general $T^i(v) \in T^i(W) \subseteq T(\overset{i}{\cdots} T(T(W))) \subseteq T(\overset{i-1}{\cdots} T(W)) \subseteq \cdots \subseteq T(W) \subseteq W$ for all $i \in \mathbb{N}$, whence $W_v \subseteq W$. Namely if W is a T-invariant subspace of V, and $v \in W$, then $W_v \subseteq W$.

<u>Theorem 60:</u> Let V be a vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation, let $W \subseteq V$ be a T-invariant vector subspace of V. Then the function $T_W : W \to W$ is well defined and it is a linear transformation.
$w \mapsto T(w)$

<u>Proof:</u> Consider $T_W : W \to W$ as given, since W is T-invariant then for all $w \in W$ we have $T(w) \in T(W) \subseteq W$ and T_W is well defined (it inputs elements in W and outputs elements in W). Moreover since T is a linear transformation then T_W is a linear transformation. $\square$.

Then T_W has a characteristic polynomial, and it is related to the characteristic polynomial of T.

Theorem 61: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be

Theorem 19: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation, let $W \leq V$ be a T-invariant vector subspace of V. Then the characteristic polynomial of T_W divides the characteristic polynomial of T.

Proof: Let $k = \dim_{\mathbb{F}}(W)$ and $\gamma = \{w_1, \dots, w_k\}$ be a basis of W. Extend γ to a basis $\beta = \{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$ of V by Corollary 18, say $n = \dim_{\mathbb{F}}(V)$. Now:

$$[T]_\beta^\beta = \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] \quad \text{with} \quad [T_W]_\gamma^\gamma = A,$$

and thus:

$$[T]_\beta^\beta - x \cdot I_n = \left[\begin{array}{c|c} A - x \cdot I_k & B \\ \hline 0 & C - x \cdot I_{(n-k)\times(n-k)} \end{array}\right] = \left[\begin{array}{c|c} [T]_\gamma^\gamma - x \cdot I_k & B \\ \hline 0 & C - x \cdot I_{(n-k)\times(n-k)} \end{array}\right]$$

Hence:

$$p_T(x) = \det\left([T]_\beta^\beta - x \cdot I_n\right) = \det(A - x \cdot I_k) \cdot \det(C - x \cdot I_{(n-k)\times(n-k)}) =$$

$$= \det\left([T]_\gamma^\gamma - x \cdot I_k\right) \cdot \det(C - x \cdot I_{(n-k)\times(n-k)}) = p_{T_W}(x) \cdot g(x)$$

where $g(x) = \det(C - x \cdot I_{(n-k)\times(n-k)})$ is a polynomial of degree $n-k$ in $\mathbb{F}[x]$. Thus $p_{T_W}(x)$ divides $p_T(x)$. $\square$.

Example: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation with eigenvalue $\lambda \in \mathbb{F}$ having associated eigenvector $v \in V$. Then $W_v = \text{span}\{v\}$ is T-invariant and $p_{T_{W_v}}(x) = \lambda - x$, which divides $p_T(x)$ because λ

being an eigenvalue means that λ is a root of $p_T(x)$. Also E_λ the eigenspace associated to λ is T-invariant and $p_{T_{E_\lambda}}(x) = (\lambda - x)^{gmult_T(\lambda)}$, which divides $p_T(x)$ because $p_T(x) = (\lambda - x)^{mult_T(\lambda)} \cdot g(x)$ and $gmult_T(\lambda) \leq mult_T(\lambda)$.

<u>Theorem 62:</u> Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation, let $v \in V$ be non-zero. Suppose that $\dim_{\mathbb{F}}(W_v) = k \in \mathbb{N}$, then:

(1) $\{v, T(v), T^2(v), \dots, T^{k-1}(v)\}$ is a basis of W_v.

(2) If $T^k(v) = a_0 \cdot v + a_1 \cdot T(v) + \cdots + a_{k-1} \cdot T^{k-1}(v)$ for some $a_0, \dots, a_{k-1} \in \mathbb{F}$ then the characteristic polynomial of T_{W_v} is:

$$p_{T_{W_v}}(x) = (-1)^k \cdot (x^k - a_{k-1} \cdot x^{k-1} - \cdots - a_1 \cdot x - a_0).$$

<u>Proof:</u> (1) Let $i \in \mathbb{N}$ be the largest natural number such that $\{v, T(v), \dots, T^i(v)\}$ is a linearly independent subset of W_v. Note that if there is no such $i \in \mathbb{N}$, namely $\{v, T(v), \dots, T^i(v), \dots\}$ is a linearly independent subset of W_v, then we found $k+1$ linearly independent vectors $v, T(v), \dots, T^k(v)$ inside W_v, contradicting that W_v has dimension k. Thus, such an $i \in \mathbb{N}$ exists. Note that since W_v is a vector subspace of V and $\{v, T(v), \dots, T^i(v)\} \subset W_v$ then $\operatorname{span}\{v, T(v), \dots, T^i(v)\} \subseteq W_v$. Moreover, let $j \in \mathbb{N}$ and consider $T^j(v)$. If $1 \leq j \leq i$ then $T^j(v) \in \{v, T(v), \dots, T^i(v)\}$

so $T^j(v) \in \text{span}\{v, T(v), \dots, T^i(v)\}$. If $j > i$ then $\{v, T(v), \dots, T^i(v)\}$ is linearly independent and $\{v, T(v), \dots, T^i(v), T^{i+1}(v)\}, \dots, \{v, T(v), \dots, T^i(v), T^{i+1}(v), \dots, T^j(v)\}$ are all linearly dependent, so by Theorem 9 we have $T^{i+1}(v) \in \text{span}\{v, T(v), \dots, T^i(v)\}$, $\dots$, $T^j(v) \in \text{span}\{v, T(v), \dots, T^i(v)\}$. Hence $T^j(v) \in \text{span}\{v, T(v), \dots, T^i(v)\}$ for all $j \in \mathbb{N}$, so $\{v, T(v), T^2(v), \dots\} \subset \text{span}\{v, T(v), \dots, T^i(v)\}$, and thus $W_v = \text{span}\{v, T(v), T^2(v), \dots\} \subseteq \text{span}\{v, T(v), \dots, T^i(v)\}$. Hence $W_v = \text{span}\{v, T(v), \dots, T^i(v)\}$. Thus $\{v, T(v), \dots, T^i(v)\}$ is a linearly independent subset of W_v that generates W_v, so it is a basis of W_v. Since W_v has dimension k then $\{v, T(v), \dots, T^i(v)\}$ must have k elements, so $i = k-1$ and thus $\{v, T(v), T^2(v), \dots, T^{k-1}(v)\}$ is a basis of W_v.

(2) Consider the basis $\beta = \{v, T(v), T^2(v), \dots, T^{k-1}(v)\}$ of W_v. Now:

$$[T_{W_v}]_\beta^\beta = \left[[T(v)]_\beta \cdots [T^{k-1}(v)]_\beta \; [T^k(v)]_\beta \right] = \begin{bmatrix} 0 & 0 & \cdots & a_0 \\ 1 & 0 & \cdots & a_1 \\ 0 & 1 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{k-1} \end{bmatrix}$$

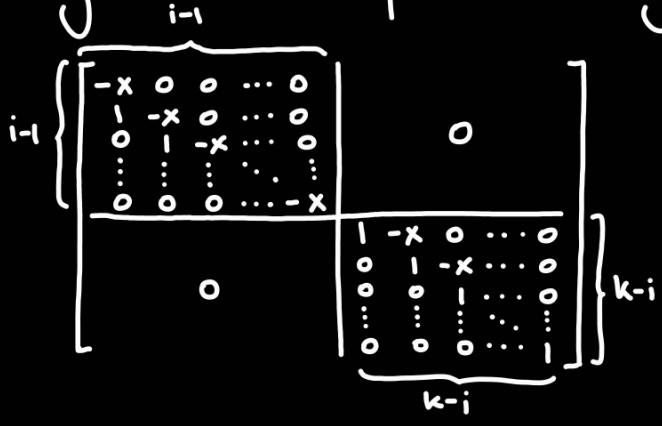
is the matrix with 1 under the diagonal and having the coefficients of the decomposition of $T^k(v)$ as a linear combination of $v, T(v), T^2(v), \dots, T^{k-1}(v)$, namely $a_0, \dots, a_{k-1}$.

Hence:

$$[T_{W_v}]_\beta^\beta - x \cdot I_k = \begin{bmatrix} -x & 0 & 0 & \cdots & 0 & a_0 \\ 1 & -x & 0 & \cdots & 0 & a_1 \\ 0 & 1 & -x & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{k-1} \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 & -x & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -x & a_{k-2} \\ 0 & 0 & 0 & \cdots & 1 & a_{k-1}-x \end{bmatrix}$$

We can expand this along the last column and use that the determinant of a triangular matrix is the product of its diagonal elements. Notice that:



is the matrix obtained from $[T_{W,v}]_{\beta}^{\beta} - x \cdot I_k$ by removing the k-th column and the i-th row, and thus:

$$\det\begin{bmatrix} -x & 0 & 0 & \cdots & 0 & \\ 1 & -x & 0 & \cdots & 0 & \\ 0 & 1 & -x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & -x & \\ \hline & & & & 1 & -x & 0 & \cdots & 0 \\ & 0 & & & 0 & 1 & -x & \cdots & 0 \\ & & & & 0 & 0 & 1 & \cdots & 0 \\ & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \det\begin{bmatrix} -x & 0 & 0 & \cdots & 0 \\ 1 & -x & 0 & \cdots & 0 \\ 0 & 1 & -x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -x \end{bmatrix} \cdot \det\begin{bmatrix} 1 & -x & 0 & \cdots & 0 \\ 0 & 1 & -x & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} =$$

$$= (-x)^{i-1}$$

Expanding $[T_{W,v}]_{\beta}^{\beta} - x \cdot I_k$ along the k-th column, when we reach coefficient a_{i-1} for $i = 1, \ldots, k-1$ we are in the i-th row, and to the determinant computed above we have to multiply by the sign $(-1)^{k+i}$ coming from the expansion of the determinant.

We then have:

$$p_{T_{W,v}}(x) = \det\left([T_{W,v}]_{\beta}^{\beta} - x \cdot I_k\right) = (a_{k-1}-x)\cdot(-x)^{k-1} + (-1)^{k+k-1} \cdot a_{k-2} \cdot (-x)^{k-2} + \cdots + (-1)^{k+(k-k)} \cdot a_0 \cdot (-x)^0 =$$

$$= (a_{k-1}-x)\cdot(-x)^{k-1} + \sum_{i=1}^{k-1} (-1)^{k+i} \cdot a_{i-1} \cdot (-x)^{i-1} = (-1)^k \cdot \left(x^k - a_{k-1} \cdot x^{k-1} - \cdots - a_1 x - a_0\right)$$

the desired result. □.

Theorem 63: (Cayley-Hamilton Theorem) Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T: V \to V$ be a linear transformation. Then $p_T(T) = 0$ in $\mathcal{L}(V,V)$.

Proof: We want to prove that $p_T(T) \in \mathcal{L}(V,V)$ is the zero linear transformation. For this, suppose $v \in V$ and consider W_v the T-cyclic subspace generated by T. Let $k = \dim_{\mathbb{F}}(W_v)$ so that $\beta = \{v, T(v), T^2(v), \dots, T^{k-1}(v)\}$ is a basis of W_v by Theorem 62. Since $T^k(v) \in W_v$ there are scalars $a_0, \dots, a_{k-1} \in \mathbb{F}$ such that $T^k(v) = a_0 \cdot v + a_1 \cdot T(v) + \cdots + a_{k-1} \cdot T^{k-1}(v)$ and thus by Theorem 62 we have $p_{T_{W_v}}(x) = (-1)^k \cdot (x^k - a_{k-1} \cdot x^{k-1} - \cdots - a_1 \cdot x - a_0)$. Note that:

$$p_{T_{W_v}}(T)(v) = (-1)^k \cdot \left(T^k(v) - a_{k-1} \cdot T^{k-1}(v) - \cdots - a_1 T(v) - a_0 v\right) = 0.$$

Moreover by Theorem 61 then $p_{T_{W_v}}(x)$ divides $p_T(x)$, namely $p_T(x) = p_{T_{W_v}}(x) \cdot g(x)$ for some $g(x) \in \mathbb{F}[x]$. Hence:

$$p_T(T)(v) = p_{T_{W_v}}(T) \cdot g(T)(v) = p_{T_{W_v}}(T)(v) \cdot g(T)(v) = 0 \cdot g(T)(v) = 0.$$

Since we can do this for every $v \in V$, we found that $p_T(T)(v) = 0$ for all $v \in V$, and thus $p_T(T): V \to V$ is the zero linear transformation. □.

Corollary 64: Let $A \in M_{n \times n}(\mathbb{F})$, then $p_A(A) = 0$ in $M_{n \times n}(\mathbb{F})$.

Corollary 64:

Proof: We have $p_A(x) = p_{T_A}(x)$ for $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^n$, $v \longmapsto Av$, and by Theorem 63 then $p_A(T_A) = p_{T_A}(T_A) = 0$. Suppose that $p_A(x) = a_0 + a_1 x + \cdots + a_n x^n$ for some $a_0, \ldots, a_n \in \mathbb{F}$, then since $\Phi : \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n) \longrightarrow M_{n\times n}(\mathbb{F})$ is an isomorphism by Theorem 41 we have:

$$0 = \Phi(0) = \Phi(p_{T_A}(T_A)) = \Phi(p_A(T_A)) = \Phi\left(a_0 + a_1 T_A + \cdots + a_n T_A^n\right) =$$

$$= a_0 + a_1 \Phi(T_A) + \cdots + a_n \Phi(T_A)^n = a_0 + a_1 [T_A]_\sigma^\sigma + \cdots + a_n \left([T_A]_\sigma^\sigma\right)^n =$$

$$= a_0 + a_1 A + \cdots + a_n A^n = p_A(A)$$

since $[T_A]_\sigma^\sigma = A$ by Theorem 29. This is the desired result. □.

For completeness, we include the relation between general invariant subspaces and the characteristic polynomial.

Theorem 65: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T : V \longrightarrow V$ be a linear transformation with $V = W_1 \oplus \cdots \oplus W_k$ for $W_1, \ldots, W_k \subseteq V$ T-invariant subspaces of V. Then $p_T(x) = p_{T_{W_1}}(x) \cdots p_{T_{W_k}}(x)$.

Proof: Use induction with the idea in the proof of Theorem 61. □.

Theorem 66: Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $T : V \longrightarrow V$ be a linear transformation with $V = W_1 \oplus \cdots \oplus W_k$ for $W_1, \ldots, W_k \subseteq V$ T-invariant

subspaces of V, let $\beta_1, \dots, \beta_k$ be basis of $W_1, \dots, W_k$ and $\beta = \beta_1 \cup \cdots \cup \beta_k$ be a basis of V.

Then $[T]_\beta = [T_{W_1}]_{\beta_1} \oplus \cdots \oplus [T_{W_k}]_{\beta_k}$.

Proof: Use induction with the matrix decomposition in the proof of Theorem 6.1. $\square$.

Here the direct sum of matrices is understood as the block matrix arising from setting them in the diagonal:

$$A = A_1 \oplus \cdots \oplus A_k = \left[\begin{array}{c|c|c|c} A_1 & 0 & \cdots & 0 \\ \hline 0 & A_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & A_k \end{array}\right].$$