Examples of linear transformations.

① Let $\mathbb{F}$ be a field, let $\mathcal{F}(\mathbb{F},\mathbb{F})$ be functions from $\mathbb{F}$ to $\mathbb{F}$.

$$\mathcal{F}(\mathbb{F},\mathbb{F}) = \left\{ f : \mathbb{F} \to \mathbb{F} \;\middle|\; \text{for one input we have exactly one output} \right\}.$$

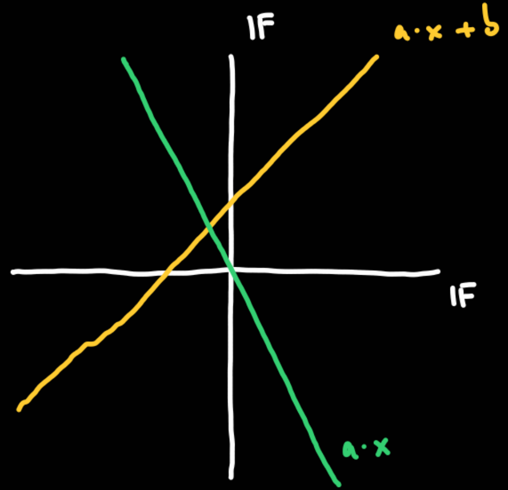
We can add functions and we can multiply functions by scalars.

$$(f+g): \mathbb{F} \longrightarrow \mathbb{F} \qquad\qquad (c\cdot f): \mathbb{F} \longrightarrow \mathbb{F}$$
$$x \longmapsto f(x)+g(x) \qquad\qquad x \longmapsto c\cdot f(x)$$

These operations make $\mathcal{F}(\mathbb{F},\mathbb{F})$ into a vector space over $\mathbb{F}$.

A linear polynomial in $\mathcal{F}(\mathbb{F},\mathbb{F})$ is determined by two scalars $a, b \in \mathbb{F}$.

Now: $\quad T: \mathbb{F} \longrightarrow \mathbb{F}$, $\; x \longmapsto a\cdot x + b$. Such a polynomial is a linear transformation when $b = 0$.



$$T(x+y) = T(x) + T(y)$$

$$T(c\cdot x) = c\cdot T(x)$$

② $V = M_{n\times m}(\mathbb{F}) \qquad W = M_{m\times n}(\mathbb{F}) \qquad T: M_{n\times m}(\mathbb{F}) \longrightarrow M_{m\times n}(\mathbb{F})$, $\; A \longmapsto A^t$

Taking the transpose of a matrix is a linear transformation.

$$T(A+B) = (A+B)^t = A^t + B^t = T(A) + T(B).$$

$$T(c\cdot A) = (c\cdot A)^t = c\cdot A^t = c\cdot T(A).$$

③ $V = \mathbb{R}_n[x]$ $\quad W = \mathbb{R}_{\underset{n-1}{n}}[x]$ $\quad T: \mathbb{R}_n[x] \longrightarrow \mathbb{R}_{\underset{n-1}{n}}[x]$

$$p(x) \longmapsto \frac{d}{dx}(p(x)) = p'(x)$$

$$T(p+q) = (p+q)' = p' + q' = T(p) + T(q)$$

$$T(c\cdot p) = (c\cdot p)' = c\cdot p' = c\cdot T(p)$$

④ $V = \mathcal{C}(\mathbb{R})$ continuous functions from $\mathbb{R}$ to $\mathbb{R}$ $\quad W = \mathcal{F}(\mathbb{R}, \mathbb{R})$

$$T: \mathcal{C}(\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R},\mathbb{R}) \qquad T: \mathcal{C}([0,1]) \longrightarrow \mathcal{F}([0,1],[0,1])$$
$$f \longmapsto \int f\cdot dx \qquad\qquad f \longmapsto \int f\cdot dx$$

$$T(f+g) = \int (f+g)\,dx = \int f\cdot dx + \int g\cdot dx = T(f) + T(g).$$

$$T(c\cdot f) = \int (c\cdot f)\,dx = c\cdot\int f\cdot dx = c\cdot T(x).$$

⑤ <u>Non-examples:</u> e^x, $\cos(x)$, $\sin(x)$, $\tanh(x)$, x^2, $\frac{1}{x}$ are <u>not</u> linear.

⑥ $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$. This is a linear transformation.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x-y \\ 2z \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \underbrace{\overset{2\times 3}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}}_{\text{applying } T \text{ is the same as left multiplication by this matrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y \\ 2z \end{bmatrix}$$

A matrix $A \in M_{m\times n}(\mathbb{R})$ is a linear transformation $A: \mathbb{R}^{\overset{\text{columns}}{n}} \longrightarrow \mathbb{R}^{\overset{\text{rows}}{m}}$.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right)$$

$$T\left(c \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c \cdot T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

Compute $\ker(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \ker(T) \Rightarrow T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x-y \\ 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x - y = 0 \text{ and } z = 0 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix}.$$

Thus $\ker(T) = \left\{ \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}$.

Compute $\operatorname{Im}(T) = \left\{ T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \Rightarrow T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 2z \end{bmatrix}.$$

If we write $\begin{bmatrix} x-y \\ 2z \end{bmatrix} = \begin{bmatrix} v \\ u \end{bmatrix}$, what are the conditions on v and u?

For all $v \in \mathbb{R}$ there are real numbers $x, y \in \mathbb{R}$ such that $x - y = v$.

For all $u \in \mathbb{R}$ there is $z \in \mathbb{R}$ such that $2 \cdot z = u$.

Explicitly: $\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} v - 0 \\ 2 \cdot \frac{u}{2} \end{bmatrix} = T\Bigg(\underbrace{\begin{bmatrix} v \\ 0 \\ u/2 \end{bmatrix}}_{\text{in } \mathbb{R}^3}\Bigg)$

Thus $\begin{bmatrix} v \\ u \end{bmatrix} \in \operatorname{Im}(T)$ for all $v \in \mathbb{R}$ and $u \in \mathbb{R}$ so $\operatorname{Im}(T) = \mathbb{R}^2$.