Proof techniques:

1. Induction.

2. Use the definition.

3. Use theorems and results.

4. Follow your nose.

1. Induction:

Show that the sum of the first n integers is $\frac{n\cdot(n+1)}{2}$.

$$1+2+\cdots+n=\frac{n\cdot(n+1)}{2}.$$

Show it's true for $n=1$: $\quad 1=\frac{1\cdot(1+1)}{2}=\frac{2}{2}$. True.

Assume the result is true for n: $\quad 1+2+\cdots+n=\frac{n\cdot(n+1)}{2}$. Induction hypothesis.

Prove the case $n+1$:

$$1+2+\cdots+n+(n+1)=\frac{(n+1)\cdot((n+1)+1)}{2}$$

We want to prove this, knowing: $1+2+\cdots+n=\frac{n\cdot(n+1)}{2}$.

$$1+2+\cdots+n+(n+1)=\frac{n\cdot(n+1)}{2}+(n+1)=\frac{n(n+1)}{2}+\frac{2(n+1)}{2}=\frac{n^2+n+2n+2}{2}=$$

use induction hypothesis: $1+2+\cdots+n=\frac{n\cdot(n+1)}{2}$.

$$=\frac{(n^2+n+n)+(n+1+1)}{2}=\frac{(n+1)\cdot(n+1+1)}{2}$$

2. Use the definition:

Given a function $f(x)$, the derivative of $f(x)$ is: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Show that for $f(x) = x^2$ we have $f'(x) = 2x$.

$$f'(x) \underset{\uparrow \text{ by definition}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + h^2 + 2xh - \cancel{x^2}}{h} =$$

$$= \lim_{h \to 0} (2x + h) = 2x.$$

Practice: The binomial coefficient is defined as: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$. $n > k$.

Show: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$. Pascal's triangle.

3. Using theorems or results.

Theorem 1: $\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$.

Theorem 2: $\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

Show that $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$.

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x) \cdot \frac{1}{g(x)}\right)' \underset{\uparrow \text{ Theorem 2}}{=} f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left(\frac{1}{g(x)}\right)' \underset{\uparrow \text{ Theorem 1}}{=}$$

$$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$$

$$= \frac{f'(x)}{g(x)} + f(x) \cdot \frac{-1}{g(x)^2} \cdot g'(x) =$$

$$= \frac{f'(x)\cdot g(x)}{g(x)^2} - \frac{f(x)\cdot g'(x)}{g(x)^2} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g(x)^2}$$

4. Follow your nose.

Show that $\sqrt{2}$ is irrational. In other words, $\sqrt{2}$ cannot be written as a fraction of integers.

Suppose $\sqrt{2} = \frac{a}{b}$ with a, b integers.

maybe even and odd integers?

Squaring this we obtain: $2 = \frac{a^2}{b^2}$. So $a^2 = 2\cdot b^2$.

We know that "even · even = even" and "odd · odd = odd".

Since a^2 is even, so a is even. Thus $a = 2\cdot k$ for k an integer.

$$a^2 = 2b^2 \quad \text{so} \quad (2\cdot k)^2 = 2b^2 \quad \text{so} \quad 4\cdot k^2 = 2b^2 \quad \text{so} \quad 2\cdot k^2 = b^2.$$

Thus b is even, by the same reasoning we used for a.

In summary, if $\sqrt{2} = \frac{a}{b}$ then a and b are both even.

However, any rational number can be written as $\frac{p}{q}$ with p, q integers, such that

irreducible fraction

p and q do not share any common divisors. $\gcd(p,q) = 1$.

If $\sqrt{2}$ were rational then it would be expressed as an irreducible fraction.

Since writing $\sqrt{2} = \frac{a}{b}$ gives that a and b share 2 as a divisor, $\sqrt{2}$ cannot be written as an irreducible fraction. Contradiction!