Math 115A
Linear Algebra

Discussion for April 25-29, 2022

Problem 1.

Let A and B be $n \times n$ invertible matrices.

(a) Prove that AB is invertible.

(b) Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Problem 2.

Let A be invertible.

(a) Prove that A^t is invertible.

(b) Prove that $(A^t)^{-1} = (A^{-1})^t$.

Problem 3.

Prove that if A is invertible and $AB = O$, then $B = O$.

Problem 4.

Let A be an $n \times n$ matrix.

(a) Suppose that $A^2 = O$. Prove that A is not invertible.

(b) Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Is A invertible? Why?

Problem 5.

(a) Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible.

(b) Let A be an $n \times m$ matrix and let B be an $m \times n$ matrix such that AB is an invertible $n \times n$ matrix. Is A invertible? Is B invertible? Why? Give examples if possible.

Problem 6($\star$).

Let A and B be $n \times n$ matrices such that $AB = I_n$.

(a) Prove that A and B are invertible.

(b) Prove that $A = B^{-1}$ and $B = A^{-1}$.

(c) State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.

We are saying that for square matrices (and for linear transformations between vector spaces of the same dimension), having a one sided inverse is equivalent to having a two sided inverse.

Problem 7.

Let A and B be matrices in $M_{n\times n}(\mathbb{F})$. We say that B is *similar* to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.

(a) Prove that $A \sim B$ whenever B is similar to A is an equivalence relation in $M_{n\times n}(\mathbb{F})$.

(b) Prove that if A and B are similar then $\mathrm{tr}(A) = \mathrm{tr}(B)$.