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Grade 1:

25% Homework

15% Discussion

25% Midterm

35% Final

Grade 2:

25% Homework

15% Discussion

60% Final

Book: Linear Algebra (4th edition) by Friedberg, Insel, Spence.

1. Fields and vector spaces

Linear algebra $\rightsquigarrow$ linear equations and linear transformations.

$\rightsquigarrow$ vector spaces and linear maps.

vector spaces: objects with structure. $+, \cdot$ (multiplication by scalars)

linear maps: functions that preserve this structure.

Definition: A field $\mathbb{F}$ is a set with sum and product:

$$+ : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F} \qquad (a,b) \longmapsto a+b$$

$$\cdot : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F} \qquad (a,b) \longmapsto a \cdot b$$

such that for all $a,b,c \in \mathbb{F}$:

(1) $a+b = b+a$ $a \cdot b = b \cdot a$

(1) $a+b = b+a \qquad a\cdot b = b\cdot a$

(2) $(a+b)+c = a+(b+c) \qquad (a\cdot b)\cdot c = a\cdot(b\cdot c)$

(3) There exists $0, 1 \in \mathbb{F}$ with $a+0=a$ and $a\cdot 1 = a$

(4) There exists $-a, \overset{a\neq 0}{a^{-1}} \in \mathbb{F}$ with $a+(-a)=0$ and $a\cdot a^{-1} = 1$.

(5) $a\cdot(b+c) = a\cdot b + a\cdot c$

$$\begin{tikzcd} (a,b,c) \in \mathbb{F}\times\underline{\mathbb{F}\times\mathbb{F}} \arrow[r, "+ \text{ on last two}"] \arrow[d, "\cdot \text{ first by the other two}"'] & \mathbb{F}\times\underline{\mathbb{F}} \quad (a, b+c) \arrow[d, "\cdot"] \\ \mathbb{F}\times\mathbb{F} \arrow[r, "+"] & \mathbb{F} \end{tikzcd}$$

(ab, ac)

The elements of $\mathbb{F}$ are called <u>scalars</u>.

<u>Examples:</u>

1. Some number sets: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ ← octonions & quaternions

2. $\mathbb{Z}$, $\mathbb{N}$

3. $\mathbb{Z}$ but we declare all even numbers to be equal
all odd numbers to be equal.

$\ldots, -2, -1, 0, 1, 2, \ldots$

$\mathbb{Z}_2 = \{\underbrace{[0]}_{\text{even}}, \underbrace{[1]}_{\text{odd}}\} \qquad 0+0\equiv 0 \qquad 0+1\equiv 1 \qquad 1+1 \equiv 2 \equiv 0$

$-1 \equiv 1$

$\mathbb{Z}_p = \{[0], [1], \ldots, [p-1]\}$

We declare that two numbers are equal if and only if they have the same reminder when divided by p.

$k \in \mathbb{Z}$ $\quad k \cdot p \equiv 0$, $k \cdot p + 1 \equiv 1$, ..., $k \cdot p + (p-1) \equiv p-1$

4. $\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \}$

<u>Definition:</u> A <u>vector space</u> V over a field $\mathbb{F}$ is a set with addition and multiplication by scalars:

$$+ : V \times V \longrightarrow V \qquad\qquad \cdot : \mathbb{F} \times V \longrightarrow V$$
$$(x, y) \longmapsto x + y \qquad\qquad (a, x) \longmapsto a \cdot x$$

such that for all $x, y, z \in V$ and $a, b \in \mathbb{F}$:

(1) $x + y = y + x$

(2) $(x + y) + z = x + (y + z)$

(3) There exists $\vec{0} \in V$ with $x + \vec{0} = x$.

(4) There exists $-x \in V$ with $x + (-x) = \vec{0}$.

(5) $1 \cdot x = x$

(6) $(\underset{\mathbb{F}}{a \cdot b}) \cdot \underset{V}{x} = a \cdot \underset{V}{(b \cdot \underset{V}{x})}$

(7) $a \cdot (x + y) = a \cdot x + a \cdot y$

(8) $(a+b)\cdot x = a\cdot x + b\cdot x$

Examples:

1. $\mathbb{R}^n = \overbrace{\mathbb{R}\times\cdots\times\mathbb{R}}^{n}$ the n-tuples of real numbers over $\overset{\mathbb{Q}}{\mathbb{R}}$.

$+$ $\qquad (\overset{\mathbb{R}}{\overset{\cup}{r_1}},\dots,\overset{\mathbb{R}}{\overset{\cup}{r_n}}) + (\overset{\mathbb{R}}{\overset{\cup}{s_1}},\dots,\overset{\mathbb{R}}{\overset{\cup}{s_n}}) = (r_1+s_1,\dots,r_n+s_n)$

$\cdot$ $\qquad a\cdot(r_1,\dots,r_n) = (a\cdot r_1,\dots,a\cdot r_n)$

Question: Is $\mathbb{R}^n$ a vector space over $\mathbb{Q}$?

Yes. $\qquad \cdot : \mathbb{Q}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$(\underset{\mathbb{R}}{\underset{\cap}{a\cdot r_1}},\dots,\underset{\mathbb{R}}{\underset{\cap}{a\cdot r_n}})$$

Question: Is $\mathbb{R}^n$ a vector space over $\mathbb{C}$?

No. $\qquad \cdot : \mathbb{C}\times\mathbb{R}^n \longrightarrow \mathbb{C}^n$

$$(\underset{\mathbb{R}}{\underset{\not\cap}{a\cdot r_1}},\dots,\underset{\mathbb{R}}{\underset{\not\cap}{a\cdot r_n}})$$

Question: Is $\mathbb{R}^n$ a vector space over any subset of $\mathbb{R}$?

superset

needs to be a field

$S \subseteq \mathbb{R}$ subset. $\qquad \mathbb{R}\subseteq S$ superset.

$\cdot : S\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$(s\cdot r_1,\dots,s\cdot r_n)$$

$S = \mathbb{Z}_2$ or $S = \mathbb{Q}$

<u>No</u>.

(or rather, be careful)

<u>Yes</u>.