Theorem 1: V vector space, $\underbrace{\text{for all}}_{\forall}$ $x, y, z \in V$, if $x + y = x + z$ then $y = z$.

Proof: We know that $x + y = x + z$. Since $x \in V$ and V is a vector space, there is $-x \in V$. $(x + (-x) = \vec{0})$

Inverse over $+$.

$$
\begin{aligned}
(x+y)+(-x) &= (x+z)+(-x) \\
&\quad \text{Associativity} \\
x+(y+(-x)) &= x+(z+(-x)) \\
&\quad \text{Commutativity} \\
x+((-x)+y) &= x+((-x)+z) \\
&\quad \text{Associativity} \\
(x+(-x))+y &= (x+(-x))+z \\
&\quad \text{Identity / Inverse} \\
y = \vec{0}+y &= \vec{0}+z = z.
\end{aligned}
$$

$\square$.

Corollary 2: V vector space. The vector $\vec{0}$ is unique.

Proof: Suppose there is $\vec{0}' \in V$ such that $z + \vec{0}' = z$ $\forall z \in V$. Now:

$$z + \vec{0} = z = z + \vec{0}'$$

so by Theorem 1 we have $\vec{0} = \vec{0}'$. $\square$.

Corollary 3: V vector space. The $\underbrace{\text{additive inverse}}_{-x}$ of x is unique.

Definition: Let V be a vector space. A **vector subspace** W of V is a subset of V if it is a vector space with the same operations as V.

Examples: $\mathbb{Q}^n \subsetneq \mathbb{R}^n \subsetneq \mathbb{C}^n$

Theorem 4: V vector space. A set W is a vector subspace of V if and only if:

$W \subseteq V$

(i) $\vec{0} \in W$.

(1) $\vec{0} \in W$.

(2) If $x, y \in W$ then $x + y \in W$.

(3) If $x \in W$ and $c \in \mathbb{F}$ then $c \cdot x \in W$.

Proof: ($\Rightarrow$) Suppose W is a vector subspace. We want to prove (1), (2), (3).

Since W is a vector subspace, it is a vector space, so it has an identity element $\vec{0}' \in W$ such that $w + \vec{0}' = w$ in W. Since W is a subset of V then $\vec{0}', w \in V$ so $w + \vec{0}' = w$ in V. Since $w = w + \vec{0}$ in V then $w + \vec{0}' = w + \vec{0}$ so by Theorem 1 then $\vec{0}' = \vec{0}$. This proves (1).

Since W is a vector space with the same operations as V then (2) and (3) are immediate.

($\Leftarrow$)

1. Commutativity: given by closure (2).

2. Associativity: (2)

3. Identity: (1)

4. Inverses: $(-1) \cdot w = -w$ * needs proof.

   then use (3).

5. Scalar identity: $\underbrace{1 \cdot w}_{\in W} = \underset{\in W}{w}$ in V, using (3)

$+ : W \times W \to W$

$\cdot : \mathbb{C} \times \mathbb{R}^n \to \cancel{\mathbb{R}}$

* $w + \underbrace{((-1) \cdot w)}_{-w} = \vec{0}$

6. Associativity of scalar multiplication: W closed under $\cdot$ by (3).

7. ——————— " ———————

8. ——————— " ——————— $\square$.

Matrices have loads of interesting subspaces.

<u>Theorem 5</u>: V vector space, U and W are vector subspaces, then $U \cap W$ is a vector subspace.

<u>Proof</u>: We only need to check (by Theorem 4):

(1) Since U and W are subspaces, then $\vec{0} \in U$ and $\vec{0} \in W$. Then $\vec{0} \in U \cap W$.

$x+y \in U \cap W$ (2) Suppose $x, y \in U \cap W$. Then $x \in U$ and $x \in W$ and $y \in U$ and $y \in W$.

Since U is a vector subspace then $x+y \in U$.

Since W is a vector subspace then $x+y \in W$. Thus $x+y \in U \cap W$.

(3) Suppose $x \in U \cap W$ and $c \in \mathbb{F}$. Then $x \in U$ and $x \in W$.

Since U is a vector subspace then $c \cdot x \in U$.

Since W is a vector subspace then $c \cdot x \in W$. Thus $c \cdot x \in U \cap W$. $\square$