Recall: V v.s. $\quad U, W \subseteq V$ v. subspaces $\quad U \cap W$ v. subspace.

Remark: $U \cup W$ is almost never a vector subspace.

Definition: V v.s. $U, W \subseteq V$ v. subspaces. The <u>sum</u> of U and W is:

$$U + W = \{ \underbrace{u + w}_{\text{in } V} \mid u \in U \text{, } w \in W \}.$$

and

&

$\wedge$

$\checkmark$

Theorem 6: $U + W$ is a vector subspace of V.

Sketch of Proof:

(1) $\vec{0} = \underset{U}{\underset{\cap}{\vec{0}}} + \underset{W}{\underset{\cap}{\vec{0}}} \in U + W$

(2) $(u + w) + (u' + w') \in U + W$

(3) $c \cdot (u + w) \in U + W$

Definition: V v.s., $U, W \subseteq V$ vector subspaces. We say that V is the <u>direct sum</u> of U and W if:

(1) $V = U + W$ and,

(2) $U \cap W = \{\vec{0}\}$.

Definition: V v.s., a vector $v \in V$ is said to be a <u>linear combination</u> of $v_1, \dots, v_n$

whenever there exist non-zero $a_1, \dots, a_n \in \mathbb{F}$ such that:

$$v = \underbrace{a_1}_{\text{coefficients}} v_1 + \cdots + \underbrace{a_n}_{\text{coefficients}} v_n .$$

Definition: V v.s., let $\{v_1, v_2, \dots\} \subseteq V$ be a subset, we define the span of $\{v_1, v_2, \dots\}$ as the set of all linear combinations of $\{v_1, v_2, \dots\}$.

$$\underbrace{\operatorname{Span}\{v_1, v_2, \dots\}}_{\substack{\text{all linear} \\ \text{combinations} \\ \text{of } \{v_1, v_2, \dots\}.}} = \left\{ a_{i_1} \cdot v_{i_1} + \cdots + a_{i_n} \cdot v_{i_n} \;\middle|\; \begin{array}{l} a_{i_1}, \dots, a_{i_n} \in \mathbb{F} \\ i_1, \dots, i_n \in \mathbb{N} \end{array} \right\}.$$

take all

$a_1 v_{①} + \cdots + a_n v_{ⓝ}$

$\vec{0} \in \operatorname{Span}\{v_1, v_2, \dots\}$

$\vec{0} = 0 \cdot v_1$ $\qquad$ $\vec{0}$ is the empty sum.

Theorem 7: $\operatorname{Span}\{v_1, v_2, \dots\} \subseteq V$ is a vector subspace.

Proof: Use Theorem 4.

(1) $\vec{0} = 0 \cdot v_1 \in \operatorname{Span}\{v_1, v_2, \dots\}$.

(2) We want two elements in $\operatorname{Span}\{v_1, v_2, \dots\}$.

finite subset of $\{v_1, v_2, \dots\}$

$$a_{i_1} v_{i_1} + \cdots + a_{i_n} v_{i_n} = \sum_{j=1}^{n} a_{i_j} \cdot v_{i_j}$$

$$\sum_{j=1}^{m} b_{i_j} \cdot v_{i_j} \qquad v_{k_j}$$

$$\sum_{j=1}^{n} a_{i_j} \cdot v_{i_j} + \sum_{j=1}^{m} b_{i_j} \cdot v_{i_j} = \sum_{j=1}^{\ell} (a_{i_j} + b_{i_j}) \cdot v_{i_j} + \sum_{j=\ell+1}^{n} a_{i_j} v_{i_j} + \sum_{j=\ell+1}^{m} b_{i_j} \cdot v_{i_j}$$

Let's say that $\{v_{i_1}, \dots, v_{i_n}\}$ coincide with $\{v_{i_1}, \dots, v_{i_m}\}$ at ⊛

$\{v_{i_1}, \dots, v_{i_\ell}\}$ for $\ell \le m, n$.

⊛ Is a finite linear combination of the desired form, so in $\mathrm{Span}\{v_1, v_2, \dots\}$

(3) $$c \cdot \left(\sum_{j=1}^{n} a_{ij} \cdot v_{i_j} \right) = \sum_{j=1}^{n} \underbrace{(c \cdot a_{ij})}_{\in \mathbb{F}} \cdot \underbrace{v_{i_j}}_{\in \{v_1, v_2, \dots\}} \in \mathrm{Span}\{v_1, v_2, \dots\}. \qquad \square.$$

__Definition:__ V v.s., $v_1, \dots, v_n \in V$, we say that $v_1, \dots, v_n$ are __linearly dependent__ whenever there exist $a_1, \dots, a_n \in \mathbb{F}$, at least one non-zero, such that:

$$a_1 v_1 + \cdots + a_n v_n = \vec{0}.$$

__Definition:__ V v.s., $v_1, \dots, v_n \in V$, we say that $v_1, \dots, v_n$ are __linearly independent__ if they are not linearly dependent.

$$a_1 v_1 + \cdots + a_n v_n \neq \vec{0} \quad \text{for all } a_1, \dots, a_n \in \mathbb{F}$$

for all   there exists ← these are logically opposite to each other.