Recall: Linear dependence/independence: $a_1 \cdot v_1 + \cdots + a_n \cdot v_n = \vec{0}$ at least one $a_i \neq 0$.

Remark: $V = \text{Span}(V)$

$$V = \text{Span}\{\underbrace{v_1, \ldots, v_n}_{\text{generators of } V}\}$$

$$V = \text{Span}\{v_1, v_2, \ldots\}$$

Theorem 8: Given $v_1, \ldots, v_n \in \mathbb{R}^n$, they are linearly independent if and only if

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$ can be row reduced to the identity.

Theorem 9: Let V be a vector space, $\{v_1, \ldots, v_n\} \subset V$ be linearly independent, $v_{n+1} \in V$.

Then $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly independent if and only if

$v_{n+1} \notin \text{Span}\{v_1, \ldots, v_n\}$.

Idea: $\{v_1, \ldots, v_n, v_{n+1}\}$ dependent $\rightsquigarrow$ $a_1 v_1 + \cdots + a_n v_n + a_{n+1} v_{n+1} = \vec{0}$.

$v_{n+1} \in \text{Span}\{v_1, \ldots, v_n\}$ $\rightsquigarrow$ $v_{n+1} = b_1 v_1 + \cdots + b_n v_n$.

$$b_1 v_1 + \cdots + b_n v_n + (-1) \cdot v_{n+1} = \vec{0}.$$

Proof: $(\Rightarrow)$ $\underbrace{\text{Suppose } \{v_1, \ldots, v_{n+1}\} \text{ is linearly independent.}}_{\text{given.}}$ Additionally, suppose

that $\underbrace{v_{n+1} \in \text{Span}\{v_1, \ldots, v_n\}}_{\text{extra}}$ to achieve a contradiction.

$v_{n+1} = a_1 v_1 + \cdots + a_n v_n$ some are not zero ...

$a_1 v_1 + \cdots + a_n \cdot v_n + (-1) \cdot v_{n+1} = \vec{0}$, so $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly dependent, a contradiction.

So the additional assumption is false, so $v_{n+1} \notin \text{Span}\{v_1, \dots, v_n\}$.

($\Leftarrow$) $\underbrace{\text{Suppose } v_{n+1} \notin \text{Span}\{v_1, \dots, v_n\}}_{\text{given}}$. We want $\{v_1, \dots, v_n, v_{n+1}\}$ is linearly independent.

Additionally, assume that $\underbrace{\{v_1, \dots, v_{n+1}\} \text{ is linearly dependent}}_{\text{extra}}$, to achieve a contradiction.

$$a_1 v_1 + \cdots + a_n v_n + a_{n+1} \cdot v_{n+1} = \vec{0} \quad \text{some } a_i \text{ not zero}$$

If $a_{n+1} = 0$ then $\underbrace{a_1 v_1 + \cdots + a_n v_n = \vec{0}}_{\text{this means that } \{v_1, \dots, v_n\} \text{ is linearly dependent.}}$. This contradicts that $\{v_1, \dots, v_n\}$ are linearly independent. Thus $\{v_1, \dots, v_{n+1}\}$ is linearly independent.

If $a_{n+1} \neq 0$ then we can rewrite:

$$v_{n+1} = \frac{-a_1}{a_{n+1}} v_1 + \cdots + \frac{-a_n}{a_{n+1}} \cdot v_n \quad \text{so} \quad v_{n+1} \in \text{Span}\{v_1, \dots, v_n\}.$$

This is a contradiction, so $\{v_1, \dots, v_{n+1}\}$ linearly independent. $\square$.

Recap: $\{-\}$ linearly independent $\Leftrightarrow$ $v_{n+1} \notin \text{Span}\{\cdots\}$.

$P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$.

$\{-\}$ linearly dependent $\Leftrightarrow$ $v_{n+1} \in \text{Span}\{\cdots\}$.

Example: $\mathbb{R}[x]$ (for all $\mathbb{F}$ infinite)

$\{1, x, x^2, \dots\}$ is linearly independent.

Uninsightful : by definition.

Insightful : $\mathbb{R}[x] \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$

<u>Idea:</u> 1. $\{1, x, \dots, x^n\}$ is linearly independent.

2. Use this for $\{1, x, \dots\}$.

$V = \text{Span}(V)$ $\quad V = \text{Span}\{v_1, \dots, v_n\}$

<u>Definition:</u> V v.s. a subset $\{v_1, v_2, \dots\} \subset V$ is a <u>basis</u> of V whenever:

(1) $V = \text{Span}\{v_1, v_2, \dots\}$.

(2) $\{v_1, v_2, \dots\}$ are linearly independent.

We say that $\{v_1, v_2, \dots\}$ form a basis of V.

<u>Example:</u> $\mathbb{R}[x] = \text{Span}\{1, x, \dots\}$

number of elements in S $\qquad$ number of elements in T

<u>Theorem 10:</u> If S and T both form a basis of V then $|S| = |T|$.

<u>Definition:</u> V v.s. with basis S. We say that V has <u>dimension</u> $|S|$.

$$\dim(V) = |S|.$$

<u>Example:</u> $\mathbb{R}^n$ has dimension n.

$\underbrace{M_{n \times m}(\mathbb{R})}_{\text{matrices with } n \text{ rows}}$ has dimension $n \cdot m$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

m columns.