Theorem 12: V v.s., $\{v_1, \dots, v_n\}$ basis of V, $v \in V$. The decomposition of v as a linear combination of the basis elements is unique.

$$a_1 v_1 + \cdots + a_n v_n = v = b_1 v_1 + \cdots + b_n v_n \quad \text{then} \quad a_1 = b_1, \dots, a_n = b_n.$$

Theorem 13: $V = \mathrm{Span}\{v_1, \dots, v_n\}$ there is a subset $S \subseteq \{v_1, \dots, v_n\}$ that is a basis of V.

Idea: Pick elements from $\{v_1, \dots, v_n\}$ making sure that what we pick is linearly independent.

Proof: If $n=0$ then $V = \mathrm{Span}\{\}$ so $V = \{\vec{0}\} = 0$. Now $\beta = \{\}$ is a basis of $V=0$. If $n=1$ then $V = \mathrm{Span}\{v_1\}$. If $v_1 = \vec{0}$ then $V = \{\vec{0}\}$ and $\beta = \{\}$ is a basis. If $v_1 \neq \vec{0}$ then $\beta = \{\vec{v}_1\}$ is a basis.

If $n \neq 0, 1$, let $V = \mathrm{Span}\{v_1, \dots, v_n\}$. Removing all the zero vectors, we can take $v_1 \neq \vec{0}$. If $\mathrm{Span}\{v_1\} = V$ then $\beta = \{v_1\}$ is a basis.

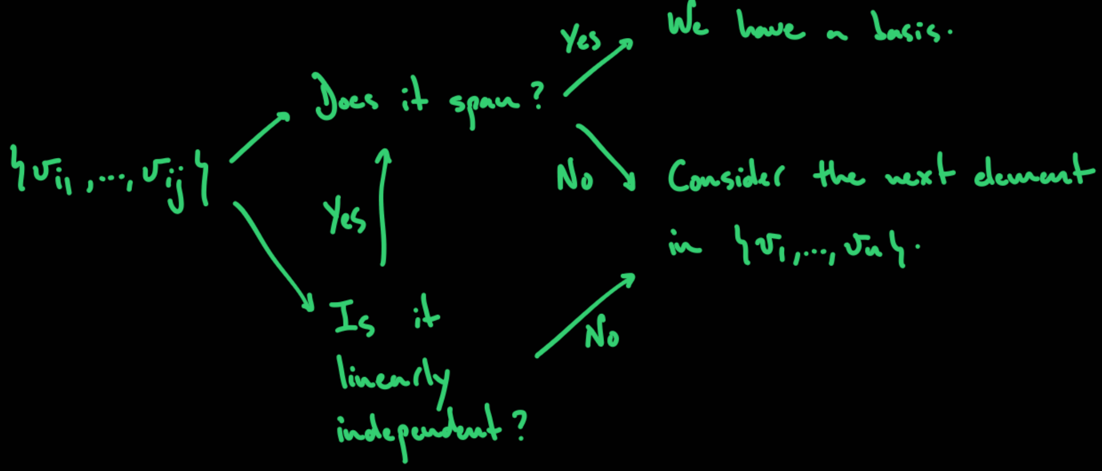
If $\mathrm{Span}\{v_1\} \neq V$, consider v_2. If $v_2 \in \mathrm{Span}\{v_1\}$, consider v_3.

If $v_2 \notin \mathrm{Span}\{v_1\}$ then $\{v_1, v_2\}$ is linearly independent by Theorem 11.

If $\mathrm{Span}\{v_1, v_2\} = V$ we are done since $\beta = \{v_1, v_2\}$ is a basis. If

$\mathrm{Span}\{v_1, v_2\} \neq V$, consider v_3

$\text{Span}\{v_1, v_2\} \neq V$ consider v_3.



Repeat this process. Since $\{v_1, \dots, v_n\}$ is finite, the process stops. We are left with $\{v_{i_1}, \dots, v_{i_k}\} \subseteq \{v_1, \dots, v_n\}$ that is linearly independent and $\text{Span}\{v_{i_1}, \dots, v_{i_k}\} = \text{Span}\{v_1, \dots, v_n\} = V$. So $\beta = \{v_{i_1}, \dots, v_{i_k}\}$ is a basis. $\square$.

<u>Remark:</u> A finite spanning set can be reduced to a basis.

If S spans V then $|S| \geq \dim(V)$.

If S is linearly independent then $|S| \leq \dim(V)$.

<u>Definition:</u> V v.s. W sub v.s. given $v \in V$ we define the <u>coset</u>

$$v + W = \{v + w \mid w \in W\}.$$

The quotient of V with W, denoted V/W is the set of sets:

$$\begin{aligned} V/W &= \{v + W \mid v \in V\} \\ &= \{\{v + w \mid w \in W\} \mid v \in V\}. \end{aligned}$$

<u>Theorem 14:</u> V/W is a vector space with:

$$+ : V/W \times V/W \longrightarrow V/W \qquad \cdot : \mathbb{F} \times V/W \longrightarrow V/W.$$
$$(v_1 + W,\ v_2 + W) \mapsto ((v_1 + v_2) + W) \qquad (c,\ v + W) \mapsto ((c \cdot v) + W)$$

<u>Theorem 15:</u> (Replacement Theorem) V v.s. if $\{v_1, \dots, v_n\}$ generates V and $\{u_1, \dots, u_m\}$ is linearly independent. Then:

1) $m \leq n$

2) There exists a subset $H = \{v_{i_1}, \dots, v_{i_{n-m}}\}$ such that:

$$\operatorname{Span}\{u_1, \dots, u_m, v_{i_1}, \dots, v_{i_{n-m}}\} = V.$$

<u>Proof:</u> On Monday.

<u>Corollary 16:</u> Every basis of V has the same number of elements.

<u>Remark:</u> 1. The cardinality of a basis <u>is</u> the dimension of V.