**Replacement Theorem:** $V = \text{Span}\{v_1, \dots, v_n\}$

$\{u_1, \dots, u_m\}$ linearly independent. Then:

1) $m \leq n$

2) $\text{Span}\{u_1, \dots, u_m, v_{i_1}, \dots, v_{i_{n-m}}\} = V$

(Sketch) **Proof:** We will use induction on $m$.

**$m=0$:** 1) $0 \leq n$ ✓ ok.

2) $\text{Span}\{v_1, \dots, v_n\} = V$. ✓ ok.

**Induction hypothesis:** fix $m$, suppose the result holds for $m-1$.

**$m$:** $\{u_1, \dots, u_m\}$ l.i.

$\{u_1, \dots, u_{m-1}\}$ l.i. By induction hypothesis:

1) $m-1 \leq n$

$V = \text{Span}\{u_1, \dots, u_{m-1}\}$ 2) $\text{Span}\{u_1, \dots, u_{m-1}, v_{i_1}, \dots, v_{i_{n-m+1}}\} = V$

If $m = n$ then we have a contradiction. (same for $m > n$)

If $m \neq n$ since $m-1 \leq n$ we have $m-1 < n$. Hence $m \leq n$.

$u_m \in V$ so: $\exists\, a_{ij} \neq 0$

$$u_m = a_1 u_1 + \cdots + a_{m-1} u_{m-1} + a_m v_{i_1} + \cdots + a_n v_{i_{n-m+1}}$$

Some $a_i \neq 0$.

(*) If $a_i = 0$ for $i \geq m$ then $u_m \in \text{Span}\{u_1, \dots, u_{m-1}\}$

but $\{u_1, \dots, u_{m-1}, u_m\}$ is l.i., a contradiction.

For $a_{i_j} \neq 0$ write:

$$-a_{i_j} \cdot v_{i_j} = a_1 u_1 + \cdots + a_{m-1} u_{m-1} - u_m + (\cdots \quad v_i \cdots)$$

$$v_{i_j} \in \underbrace{\text{Span}\{\overbrace{u_1, \dots, u_{m-1}, u_m}^{m}, \overbrace{v_{i_1}, \dots, v_{i_{j-1}}, v_{i_{j+1}}, \dots, v_{i_{n-m+1}}}^{n-m}\}}_{\text{Span}\{u_1, \dots, u_{m-1}, v_{i_1}, \dots, v_{i_{n-m+1}}\} = V}$$

$\square$.

**Theorem 17:** $\underset{\text{v.s.}}{W} \subseteq \underset{\text{v.s.}}{V}$, $V$ finite dimensional. Then:

1) $\dim(W) \leq \dim(V)$

2) $\dim(W) = \dim(V)$ if and only if $W = V$.

**Corollary 18:** Let $\underset{\text{v.s.}}{W} \subseteq \underset{\text{v.s.}}{V}$, let $\beta$ be a basis of $W$. We can add elements to $\beta$ to make it a basis of $V$.

## 2. Linear transformations.

**Definition:** $T: \underset{+}{V} \to \underset{+}{W}$ is called a linear transformation when:

(1) $T(x+y) = T(x) + T(y)$

(2) $T(c\cdot x) = c\cdot T(x)$

This always makes sense if $V$ and $W$ have the same base field $\mathbb{F}$.

__Theorem 19:__ Let $\mathcal{L}(V,W)$ be the set of all the linear transformations from $V$ to $W$. This is a vector space.

$\longrightarrow$ between sets

$\longmapsto$ between elements

$$+ : \mathcal{L}(V,W) \times \mathcal{L}(V,W) \longrightarrow \mathcal{L}(V,W)$$
$$(T_1, T_2) \longmapsto \underbrace{T_1 + T_2 : V \to W, \quad x \mapsto T_1(x) + T_2(x)}_{\text{we have to prove that } T_1 + T_2 \text{ is a linear transformation.}}$$

$A \times B =$
$= \{(a,b) \mid a \in A, b \in B\}$

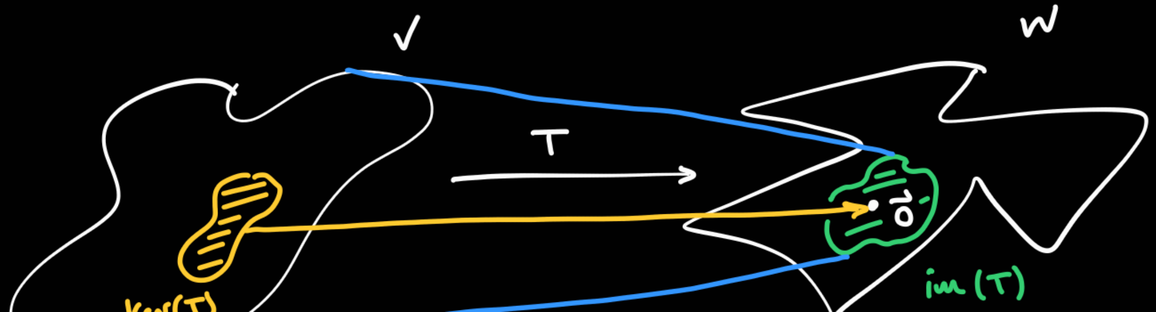
base field of $V$ and $W$

$$\cdot : \mathbb{F} \times \mathcal{L}(V,W) \longrightarrow \mathcal{L}(V,W)$$
$$(c, T) \longmapsto \underbrace{c\cdot T : V \to W, \quad x \mapsto c\cdot T(x)}_{\text{we have to prove that } c\cdot T \text{ is a linear transformation.}}$$

__Definition:__ $T: V \to W$ linear transformation:

$\ker(T) = \{x \in V \mid T(x) = \vec{0}\}$ the __kernel__ of $T$.

$\operatorname{im}(T) = \{y \in W \mid \exists\, x \in V \text{ with } T(x) = y\}$ the __image__ of $T$.



**Theorem 20:** $T: V \to W$ linear transformation.

$\mathrm{im}(T)$ is a vector subspace of $W$

$\ker(T)$ is a vector subspace of $V$

**Theorem 21:** Let $\beta = \{v_1, \dots, v_n\}$ be a basis of $V$. Then:

$T(\beta) = \{T(v_1), \dots, T(v_n)\}$ spans $\mathrm{im}(T)$.

$T: V \to W$

$x \in V$

$$T(x + (-x)) = T(x) + T((-x)) = T(x) + (-1)\cdot T(x) = T(x) - T(x) = \vec{0}$$

so $\vec{0} \in \mathrm{im}(T)$.