<u>Recall:</u> $T: V \longrightarrow W$ $\quad \ker(T) \subseteq V \quad \operatorname{im}(T) \subseteq W$

injective $\qquad$ surjective

<u>Theorem 24:</u> $T: V \longrightarrow W$ linear transformation, let $\dim(V) = \dim(W)$ be finite.

Then the following are equivalent:

(1) T is injective.

$\Downarrow$

(2) T is surjective.

$\Downarrow$

(3) $\dim(\operatorname{im}(T)) = \dim(V)$.

<u>Proof:</u> We will prove (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (3) T is injective. By the Rank-Nullity Theorem:

$$\dim(V) = \dim(\operatorname{im}(T)) + \dim(\underbrace{\ker(T)}_{\{\vec{0}\} \text{ because } T \text{ injective.}}).$$

$$\dim(V) = \dim(\operatorname{im}(T)).$$

(3) $\Rightarrow$ (1) If $\dim(V) = \dim(\operatorname{im}(T))$ then by the Rank-Nullity Theorem

$\dim(\ker(T)) = 0$ so $\ker(T) = \{\vec{0}\}$ so T is injective.

(2) $\Rightarrow$ (3) T surjective so $\operatorname{im}(T) = W$.

$$\dim(\operatorname{im}(T)) = \dim(W) = \dim(V).$$

(3) $\Rightarrow$ (2) If $\dim(V) = \dim(\operatorname{im}(T))$ then ...

$(3) \Rightarrow (2)$ If $\dim(V) = \dim(\mathrm{im}(T))$ then:

$$\dim(W) = \dim(V) = \dim(\mathrm{im}(T)) \quad \text{we have} \quad \mathrm{im}(T) \subseteq W$$

so $\mathrm{im}(T) = W$ so T surjective. $\square$.

<u>Remark:</u> We should often consider computing dimensions of $\ker(T)$ and $\mathrm{im}(T)$ instead of the dimensions of V and W.

<u>Remark:</u> When checking anything about a linear transformation $T: V \longrightarrow W$, it is enough to do so on a basis of V.

<u>Example:</u> $T: \underbrace{\mathbb{R}_2[x]}_{P_2(\mathbb{R})} \longrightarrow \underbrace{\mathbb{R}_3[x]}_{P_3(\mathbb{R})}$, check if T is injective or surjective.

$$f(x) \longmapsto 2f'(x) + \int_0^x 3f(x)\,dx$$

Method: compute $\dim(\ker(T))$ and $\dim(\mathrm{im}(T))$.

$\mathbb{R}_2[x] = \mathrm{Span}\{\underbrace{1, x, x^2}_{\text{basis}}\}$ $\quad T(1) = 3x \quad T(x) = 2 + \frac{3x^2}{2} \quad T(x^2) = 4x + x^3$

So $\mathrm{im}(T) = \mathrm{Span}\{\underbrace{3x, 2 + \frac{3x^2}{2}, 4x + x^3}_{\text{linearly independent}}\}$ so $\dim(\mathrm{im}(T)) = 3$.

Since $\mathbb{R}_3[x]$ has dimension 4, T is not surjective.

By Rank-Nullity Theorem:

$$\underbrace{\dim(\mathbb{R}_2[x])}_{3} = \dim(\ker(T)) + \underbrace{\dim(\mathrm{im}(T))}_{3} \quad \text{so} \quad \dim(\ker(T)) = 0.$$

So T is injective.

Theorem 25: $T: V \to W$, let $\{v_1, \dots, v_n\}$ be a basis of V, let $\{w_1, \dots, w_n\}$ be a basis of W. Suppose that $T(v_i) = w_i$ for all $i = 1, \dots, n$. Then T is unique.

Proof: Let $T': V \to W$ be a linear transformation such that $T'(v_i) = w_i$ for all $i = 1, \dots, n$. We want to prove $T = T'$.

Recall that two functions are the same when they send the same element in the source to the same element in the image.

Pick $v \in V$, since $\{v_1, \dots, v_n\}$ is a basis of V, write $v = \sum_{i=1}^{n} a_i \cdot v_i$.

Now: $$T(v) = T\left(\sum_{i=1}^{n} a_i \cdot v_i\right) = \sum_{i=1}^{n} a_i \cdot \underbrace{T(v_i)} = \sum_{i=1}^{n} a_i \cdot w_i =$$

$$= \sum_{i=1}^{n} a_i \cdot \underbrace{T'(v_i)} = T'\left(\sum_{i=1}^{n} a_i \cdot v_i\right) = T'(v).$$

So $T = T'$. $\qquad T(v_i) = T'(v_i)$ $\qquad \square$.

$v = \sum_{i=1}^{n} a_i \cdot v_i$.

Definition: Let V be a vector space with basis $\beta = \{v_1, \dots, v_n\}$. Suppose that $v \in V$ is expressed as $v = \sum_{i=1}^{n} a_i \cdot v_i$. We say that $a_1, \dots, a_n$ are the <u>coordinates</u> of v with respect to β. The <u>coordinate vector</u> of v with respect to β is:

$$[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \neq \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_\beta$$

<u>Example:</u> $\mathbb{R}_2[x]$ $\qquad \beta_1 = \{1, x, x^2\}$ $\qquad \beta_2 = \{1+x, 1-x, 3x^2\}$

$$p(x) = 3 - 2x + 4x^2 \qquad [p(x)]_{\beta_1} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \qquad [p(x)]_{\beta_2} = \begin{bmatrix} 1/2 \\ 5/2 \\ 4/3 \end{bmatrix}.$$

$$\parallel$$

$$p(x) = \frac{1}{2}(1+x) + \frac{5}{2}(1-x) + \frac{4}{3}3x^2$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S = \{e_1, e_2, e_3, e_4\}$$